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MAA

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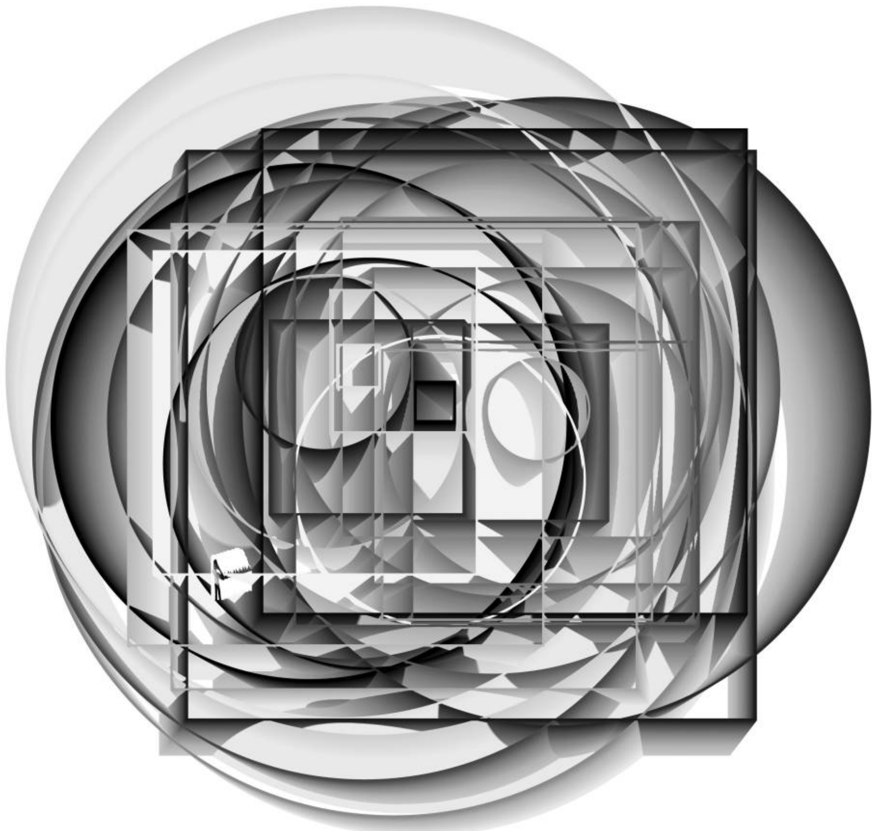

"DRAWING_2013_10_29_1_2_18"

- Bernoulli's *Ars Conjectandi*
- Charles Hugo Kummell, statistician
- The critical points are the foci, by linear algebra

Math 2013

An Official Publication of the **Mathematical Association of America**

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Mathematics Magazine aims to provide lively and appealing mathematical exposition. The *Magazine* is not a research journal, so the terse style appropriate for such a journal (lemma-theorem-proof-corollary) is not appropriate for the *Magazine*. Articles should include examples, applications, historical background, and illustrations, where appropriate. They should be attractive and accessible to undergraduates and would, ideally, be helpful in supplementing undergraduate courses or in stimulating student investigations. Manuscripts on history are especially welcome, as are those showing relationships among various branches of mathematics and between mathematics and other disciplines.

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MAA

MATHEMATICS MAGAZINE

LETTER FROM THE EDITOR

We are pleased to announce that Mike Jones will be the next Editor of the MAGAZINE, for the term 2015–2019. New manuscripts submitted during 2014 will go first to Mike. The submission process is changing; read about it on page 399.

This year is the 300th anniversary of the publication of Jacob Bernoulli's *Ars Conjectandi*, which had a considerable influence in combinatorics and probability. Our opening article, by Gerald L. Alexanderson, puts a famous result from the book in context.

One way to understand history is through the careers of individual mathematicians. Perhaps some of us see Bernoulli as a role model, but I suspect more of us will identify with Charles Kummell, whose story is told in the article by Asta Shomberg and James Tattersall. Kummell was a government statistician who worked in Detroit from 1871 to 1880, and then in Washington, DC, until 1897. He wrote foundational papers on least-squares estimation, treating problems that arose directly in his government work. Like many of our readers, he was an immigrant. Like a few readers, he published his first paper at the age of 40.

When Kummell moved to Washington, he found an active local scientific society, but for most of his career there were no national mathematical organizations and few American mathematics journals. As far as professional organizations go, we live in a golden age!

The article by Clifford and Lachance begins with the Bôcher–Grace Theorem—also called Marden's Theorem—which is about ellipses and cubic equations. They have found an elegant extension of the theorem to parallelograms and fourth-degree equations, and they prove it using basic linear algebra.

Some of the articles in this issue find unifying themes in their subject matter. Alessandro Fonda presents a simple sum-of-squares formula that unites several theorems about triangles, parallelograms, and tetrahedra. Vincent Matsko starts with a familiar image that may remind you of the "sliding ladder" problem, and generalizes it to other ellipse-like figures. Did you know that every fifth Fibonacci number is divisible by 5? Does something like that happen for other primes? Other recursions? Marc Renault tells that story in his note. Mathes and Dupree start with an enumeration of the rationals, and turn it into a function with a dense graph.

We have a crossword puzzle! If you enjoy it, let us know. Copies of the puzzle are very easy to download, so you don't have to write in the library copy of the MAGAZINE.

As always in December, we are pleased to acknowledge the work of our referees. Many talented people contribute to the preparation and production of the MAGAZINE, as to all of the MAA journals.

Walter Stromquist, Editor

ARTICLES

An Anniversary for Bernoulli's *Ars Conjectandi*

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The year 2013 marks the tercentenary of the appearance of Jacob (Jacques) Bernoulli's posthumous masterpiece, his *Ars Conjectandi*, which appeared in 1713, eight years after Bernoulli's death. Jacob was one member of that vast family of Bernoullis in Basel—rather the mathematical equivalent of the large number of Bachs in Leipzig—active at roughly the same time. The author of the *Ars Conjectandi* was probably the most illustrious member of the family, but he had a close competitor in his younger brother Johann (Jean). There were many others, most notably Daniel, who worked primarily in St. Petersburg, but the family, frugal at coming up with new first names, produced enough Jacobs, Johanns, Nikolauses, Christoffs, and Daniels that it helps to give them the equivalent of subscripts. Henceforth "Bernoulli" here will refer, unless otherwise noted, to Jacob I Bernoulli.

Among works of members of the family, the *Ars Conjectandi* is in a class by itself, the only work by any of the Bernoullis to make it into the Grolier Club volume, Harrison Horblit's *One Hundred Books Famous in Science* of 1964, where it is described as "the establishment of the fundamental principles of the calculus of probabilities." This is not the first time an anniversary of the book has been celebrated. The "Law of Large Numbers" that appeared in the *Ars Conjectandi* was recognized in a conference at the St. Petersburg Academy of Sciences in December of 1913. Manfred Denker [**4**, 373] quotes from A. A. Markov's speech for that conference: "More than two hundred years have passed since Bernoulli's death but he lives and will live in his theorem." This echoes the inscription on a plaque honoring Bernoulli and accompanying a logarithmic spiral: *Eadem mutata resurgo* (though changed I will arise the same). It is in the Basel cathedral and presumably refers to the soul of Bernoulli. Denker's substantial survey article covers the impact of the *Ars Conjectandi* right up to the present time.

It is widely reported, and with ample documentation, that Jacob and Johann Bernoulli did

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not have a warm relationship. Instead of combining their obvious mathematical powers and demonstrating filial loyalty, they spent their lives competing against each other, all the way down to professional rivalry over the chair in mathematics at the University of Basel, where Johann did not succeed to the chair until his brother, who occupied it for many years, died.

The *Ars Conjectandi* begins with a discussion of permutations and combinations, repeating to some extent earlier work of Pascal and Huygens. It was in the second part that he derived formulas for the sums of powers of the positive integers. These are interesting in their own right but more important because in developing these formulas recursively he introduced the Bernoulli numbers, which appear in other interesting contexts in analysis. In the third and fourth sections he investigated the "Law of Large Numbers" with ramifications in the theory of probability. This work was carried out between two contributions in this area by Abraham De Moivre, an early monograph and De Moivre's book, his *Doctrine of Chances*, which appeared in 1718. In that classic book De Moivre already referred to the contributions of Bernoulli.

Here we'll examine a partial translation of the *Ars Conjectandi* into English, included in a larger volume by Francis Maseres, identified as "Cursitor Baron of the Court of Exchequer." It was published in 1795. The contents are described on the title page as:

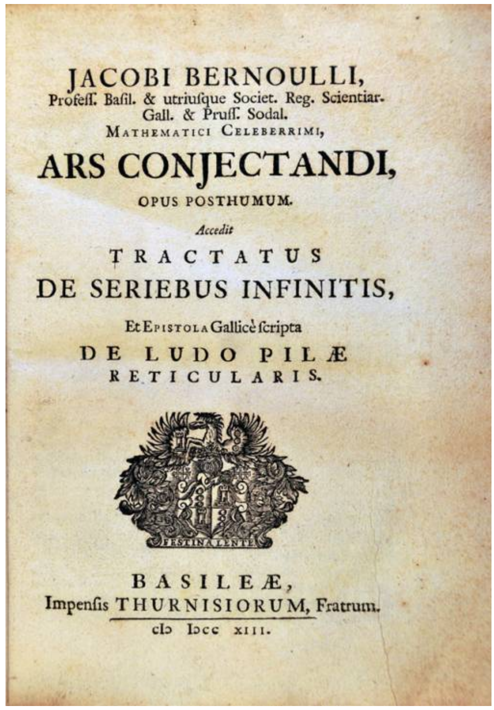
> "The Doctrine of Permutations and Combinations, Being an Essential and Fundamental Part of the Doctrine of Chances; As it is delivered by Mr. James Bernoulli, in his excellent Treatise on the Doctrine of Chances, intitled [sic], *Ars Conjectandi* , and by the celebrated Dr. John Wallis, of Oxford, in a Tract intitled from the Subject, and published at the end of his Treatise on Algebra: In the former of which Tracts is contained, A Demonstration of Sir Isaac Newton's famous Binomial Theorem, in the Cases of Integral Powers, and of the Reciprocals of Integral Powers, together with some other useful Mathematical Tracts."

It is a cluttered title page.

Along with the English translation, Maseres reprints the original Latin edition but only the first three chapters of the second "book." At the end of the excerpt from the *Ars Conjectandi*, Bernoulli derives recursive formulas for finding the sums of the kth powers of the first n positive integers, up to $k = 10$ (a subject of interest in any calculus class when asked to find the value of some definite integrals without benefit of the Fundamental Theorem.) This problem had attracted attention previously. At this point in the *Ars Conjectandi* we find a hint of Jacob's concern that he should get proper credit for what he had done. He writes (taken from Maseres's translation):

> "I cannot but observe on this occasion, that the learned Ismael Bullialdus, or Bouillaud, has been rather unfortunate in his manner of treating this subject, in his Treatise on the Arithmetick of Infinites; since the whole of the folio volume which he has written upon it does nothing more than enable us to find the sums of the first six powers of the natural numbers, 1, 2, 3, 4, 5, 6, 7, &c, continued to any given number n; which is only a part of what we have here accomplished in the compass of a dozen pages." [**11**, 197].

Maseres points out in a footnote that an account of Bouillaud's work appears in chapter 80 of John Wallis's *Treatise of Algebra* [**17**, 310–311]. The problem no longer appears to be challenging. A proliferation of recursive methods for finding these formulas has developed into a veritable cottage industry over the years; see [**1, 2, 5, 6, 8,**



JACOBI BERNOULLI,
Profeſſ. Baſil. & utriuſque Societ. Reg. Scientiar.
Gall. & Pruſſ. Sodal.
MATHEMATICI CELEBERRIMI,

ARS CONJECTANDI,
OPUS POSTHUMUM.
Accedit
TRACTATUS
DE SERIEBUS INFINITIS,
Et EPISTOLA Gallicè ſcripta
DE LUDO PILÆ
RETICULARIS.

BASILEÆ,
Impenſis THURNISIORUM, Fratrum.
cIↃ IↃcc xIII.

9, 10, 12, 15]. A parallel series of papers on this problem over this period has appeared in the British *Mathematical Gazette*. The problem has even reached the stage of being an exercise for students in George Pólya's *Mathematical Discovery* [**16**, 66–68].

Bernoulli was rather dismissive of Bouillaud's contribution to this problem, appearing in 1682, but Bouillaud was a respected French astronomer-priest and friend of Huygens, Mersenne, and Pascal. Newton claimed that Bouillaud's calculations of the sizes of the orbits of the planets were, along with Kepler's, the most accurate then available. More impressive than Bernoulli's extension of the results all the way up to $k = 10$ (with great economy, as he asserts) is the way Bernoulli dazzles us with the computation of the sum of all the tenth powers of the first thousand natural numbers:

$$91, 409, 924, 241, 424, 243, 424, 241, 924, 242, 500,$$

which he points out [in British usage] is "more than 91 quintillions, or 91 times the fifth power of a million" [**3**, 98]. That probably counted as a big number in 1713.

What Bernoulli apparently did not know was that even before Bouillaud's work, Johann Faulhaber, a teacher in Ulm, had published in his *Academia Algebræ* the formulas up to $k = 17$ in 1631. Details of Faulhaber's methods appear in [**10**]. More startling is the fact that Faulhaber claimed to have shown even more, though his arguments are not entirely clear: For odd values of k, the sum of the powers is a polynomial in $n(n+1)$ and in the case of even k, a polynomial in $n(n+1)$ times $2n+1$. These days this demonstration can be carried out with arguments about Eulerian polynomials, but this pattern is suggested even by the first few examples if we let $N = n(n+1)/2$, for $k = 1$, the sum is N; $k = 2$: $N(2n+1)/3$; $k = 3$: N^2; and $k = 4$: $N(6N-1)(2n+1)/15$.

Donald Knuth, in a long and fascinating paper dedicated to the memory of D. H. Lehmer [**9**], noted that these claims by Faulhaber were first proved by C. G. J. Jacobi in 1834 [**7**]. Knuth wrote down the formula for the 17th powers, a 9th degree polynomial in N, where $N = n(n+1)/2$, that is, an 18th degree polynomial in n. It was a good piece of calculating for 1631. Knuth also points out, however, that at that time it was not unusual for mathematicians to write down their results in secret code to protect the results from rivals. At the end of his piece Faulhaber suggested that he had a formula for the 25th powers, but he did not write it down. Instead he posed a riddle involving the letters J E S U, presumably a reference to Jesus. Knuth tried to crack the code to reconstruct the formula but was unsuccessful. He did succeed, however, in retrieving the sum for the 23rd powers. Not bad! What might Bernoulli have done with this had he been aware of Faulhaber's results?

REFERENCES

1. Dumitru Acu, Some algorithms for the sums of integer powers, *Math. Mag.* **61** (1988) 189–192.
2. Alan F. Beardon, Sums of powers of integers, *Amer. Math. Monthly* **107** (1996) 201–213.
3. Jacob Bernoulli, *Ars Conjectandi,* Thurnisiorum, Fratrum, Basel, 1713.
4. Manfred Denker, Tercentennial anniversary of Bernoulli's Law of Large Numbers, *Bull. Amer. Math. Soc.* **50** (2013) 373–390.
5. A. W. F. Edwards, A quick route to sums of powers, *Amer. Math. Monthly* **93** (1986) 451–455.
6. Ira Gessel and University of South Alabama Problem Group, A formula for power sums, *Amer. Math. Monthly* **95** (1988) 961–962.
7. C. G. J. Jacobi, De usu legitimo formulae summatoriae Maclaurinianae, *J. Reine Angew. Math.* **12** (1834) 263–272.
8. Clive Kelly, An algorithm for sums of integer powers, *Math. Mag.* **57** (1984) 296–297.
9. Donald E. Knuth, Johann Faulhaber and sums of powers, *Math. Comp.* **61** (1993) 277–294.
10. H. K. Krishnapriyan, Eulerian polynomials and Faulhaber's result on sums of powers of integers, *College Math. J.* **26** (1995) 118–123.
11. Francis Maseres, *The Doctrine of Permutations and Combinations, Being an Essential and Fundamental Part of the Doctrine of Chances,* B. and J. White, London, 1795.
12. Robert W. Owens, Sums of powers of integers, *Math. Mag.* **65** (1992) 38–40.
13. J. L. Paul, On the sum of the kth powers of the first n integers, *Amer. Math. Monthly* **78** (1971) 271–272.
14. David E. Penney and Carl B. Pomerance, Multiplicative relations for sums of initial kth powers, *Amer. Math. Monthly* **92** (1985) 729–731.
15. Pedro A. Piza, Powers of sums and sums of powers, *Math. Mag.* **25** (1952) 137–142.
16. George Pólya, *Mathematical Discovery*, vol. 1, New York, John Wiley, 1967.
17. John Wallis, *Treatise of Algebra,* John Playford, London, 1685.

Summary An exposition of Jacques Bernoulli's contribution to an oft-visited problem: finding formulas for the sum of the kth powers of the first n positive integers, formulas seen in beginning calculus courses. Bernoulli published his results in his masterpiece, *Ars Conjectandi,* published in 1713, a famous work which we celebrate in this, its tercentenary year.

GERALD L. ALEXANDERSON is a former editor of *Mathematics Magazine* (1986–1990) and during his 59 years of membership in the MAA, 24 of those years he was a member of the MAA's Board of Governors. He has also served as First Vice President (1984–1986), Secretary (1990-1997), President (1997–1999), and over his years as a member he served on 68 MAA committees. He survived, even to the point of also having a life outside the MAA, as Valeriote Professor of Science at Santa Clara University.

Life and Statistical Legacy of Charles Hugo Kummell

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According to Stephen Stigler [**23**], European statistical research flourished in the years 1770 to 1850, while statistical research in the United States did not develop fully until after 1850, when westward migration increased the need for reliable maps and hydrographic charts. The establishment of the United States Coast Survey, the Lake Survey, and the Nautical Almanac in 1807, 1841, and 1849, respectively, set the stage for the rapid advancement of interest in statistics and the development of statistical methods. These methods were pioneered by Robert Adrain, Benjamin Peirce, his son Charles Sanders Peirce, Simon Newcomb, and Erastus Lyman De Forest. Stigler described their statistical accomplishments, but admitted that his account was incomplete without the work of C. H. Kummell and R. J. Adcock.

In this article, we focus on the achievements of Charles Hugo Kummell, who was a statistician for the Lake Survey and the U.S. Coast and Geodetic Survey. We describe his research into laws of errors and his contributions to the development of the least-squares method, as well as his efforts to promote the subject of statistics as a member of the Philosophical Society of Washington.

Life and work

Gottfried Wilhelm Hugo Karl Kummell was born August 26, 1836 in the town of Wetter Kurfürstenthum, in the Prussian Electorate of Hesse-Cassel. He was home schooled by his father, an attorney, and by private tutors until age fourteen, and then attended the Polytechnique School at Cassel. Kummell entered the University of Marburg (now the Philipps-Universität Marburg) on May 5, 1852 as an engineering student with a strong interest in classical music. He remained at the university until the winter term 1853–54, when he left without taking a degree. Kummell taught in Prussia for several years, and in the summer of 1866 he left for Norfolk, Virginia, to live with his older sister. Upon arriving in America, he changed his name to Charles Hugo Kummell. For the next five years, he taught mathematics and music in Norfolk.

On April 25, 1871, Kummell was appointed an assistant engineer with the U.S. Lake Survey and moved to its headquarters in Detroit, Michigan. The Lake Survey had been created by Congress in 1841. The agency was charged with conducting hydrographical surveys of the northern and northwestern lakes, and preparing nautical charts and other navigation aids. It published its first maps in 1852.

While in Detroit, Kummell married Anna Wackwitz and the couple had two children, Matilda (1876) and Frederick August (1877). Kummell's work at the Survey consisted mostly of telegraphic determination of differences of longitudes, which at that time was one of the most accurate ways to determine terrestrial longitude. Latitudes

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can be determined independently, but longitudes can only be determined relative to other longitudes. Given two locations, their difference in longitude can be determined by comparing local times, and the telegraph is an effective means of synchronization.

In 1876, the year he turned forty, Kummell published his first paper, "New Investigations of the Law of Errors of Observations." Later, he published over two dozen articles, whose topics varied from error analysis to differential geometry, abstract algebra, geometry, astronomy, and physics.

By 1882, the Lake Survey had completed its original mandate. Kummell left the agency on October 31, 1880, and on November 8 he accepted a position as a computer with the U.S. Coast and Geodetic Survey in Washington, D.C. During his tenure there, Kummell attended to geodetic computations for geographical positions of secondary and tertiary stations located in the West Coast and in New England. This included revising abstracts of vertical angles necessary to determine the precise elevations of primary stations and adjustment of secondary and tertiary triangulations. The Coast Survey maintained a number of magnetic stations that recorded the fluctuation and intensity of the earth's magnetic field. Kummell worked on more accurate methods for determining diurnal and secular changes in the earth's magnetism and the direction of the magnetic poles at the stations. The new methods considerably reduced the error in compass measurements and led to the construction of more accurate charts and land surveys. There is no evidence that Kummell ever engaged in any field work or ever left Washington for any extended period of time. Kummell remained with the USCGS until his death in 1897.

Least-squares solutions

Kummell's work in Washington consisted largely of determining and solving systems of normal equations. He noted that "upon close examination it will be found that a great many observations are of a more or less complex nature so that the observational equations contain more than one observed quantity, and it may be of interest to show how such observational equations should be treated [**8**]."

To gain some insight into his daily work, consider the following problem. Suppose that there are n observational equations of m unknowns $\{X_1, X_2, \ldots, X_m\}$,

$$\begin{aligned} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1m}X_m &= b_1 \\ a_{21}X_1 + a_{22}X_2 + \cdots + a_{2m}X_m &= b_2 \\ &\vdots \\ a_{n1}X_1 + a_{n2}X_2 + \cdots + a_{nm}X_m &= b_n, \end{aligned} \tag{1}$$

where $n > m$. The b_i are observed values, perhaps of heights, distances, or angles, and the a_{ij} are known values. Since the number of equations exceeds that of unknowns, the above system is not likely to have an exact solution. Kummell employed the method of least squares to determine the "most probable" values for the m unknowns. Suppose that $\{x_1, x_2, \ldots, x_m\}$ are the most probable values of $\{X_1, X_2, \ldots, X_m\}$. Then

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m - b_1 &= r_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m - b_2 &= r_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m - b_n &= r_n, \end{aligned} \tag{2}$$

where r_i are the residuals. According to the law of errors, the most probable values of the unknown quantities are those that minimize $\sum_{i=1}^{n} r_i^2$.

To determine the most probable value of X_1, let

$$M_i = a_{i2}x_2 + \cdots + a_{im}x_m - b_i,$$

where $1 \le i \le n$. Then the first equation in (2) can be rewritten as $r_1 = a_{11}x_1 + M_1$, and the other equations can be rewritten similarly. Thus,

$$\sum_{i=1}^{n} r_i^2 = \sum_{i=1}^{n} (a_{i1}x_1 + M_i)^2 .$$

Setting the derivative of this expression with respect to x_1 equal to zero and rearranging terms yields the following *normal equation*:

$$\left(a_{11}^2 + a_{21}^2 + \cdots + a_{n1}^2\right) x_1 + a_{11}M_1 + a_{21}M_2 + \cdots + a_{n1}M_n = 0.$$

The equation can be rewritten as

$$c_{11}x_1 + c_{12}x_2 + \cdots + c_{1m}x_m = d_1,$$

where

$$d_1 = a_{11}b_1 + a_{21}b_2 + \cdots + a_{n1}b_n.$$

This sequence of operations is repeated for $x_2, x_3, \ldots, x_m$, to obtain a set of m normal equations in m unknowns:

$$\begin{aligned} c_{11}x_1 + c_{12}x_2 + \cdots + c_{1m}x_m &= d_1 \\ c_{21}x_1 + c_{22}x_2 + \cdots + c_{2m}x_m &= d_2 \\ &\vdots \\ c_{m1}x_1 + c_{m2}x_2 + \cdots + c_{mm}x_m &= d_m, \end{aligned}$$

whose solutions are the most probable values of $\{X_1, X_2, \ldots, X_m\}$.

This is called the *least-squares solution* to the overdetermined system (1), and it is a central idea of linear regression. A modern description might use matrix notation, so that equation (1) might be written as $AX = B$ and the normal equations as $A^TAx = A^TB$.

In Kummell's day, the process was laborious and time consuming, and required enormous concentration. Kummell was the man for the task. One of his peers described his work as "characterized by neatness, thoroughness, and practical application. Accustomed to computation, he set forth his results with a fullness of example well adapted to the computer's needs [**2**]."

In 1894, together with B. C. Washington, the District of Columbia correspondent for the *Electrical World and Engineering*, Kummell performed the bulk of the calculations for the forty tables in Robert Simpson Woodward's *Smithsonian Geographical Tables*. Commenting on Kummell's and Washington's work, Woodward, a professor of mechanics at Columbia, remarked that it was completed with "skill and fidelity and it is believed that the systematic checks supplied by them have rendered the tables they computed entirely trustworthy [**25**]."

When Kummell joined the Coast and Geodetic Survey, the post of Superintendent was held by Carlisle P. Patterson, the successor to Benjamin Peirce. Patterson died in 1881, and was replaced by Julius Erasmus Hilgard, a native of Bavaria who had studied engineering in Philadelphia. From 1850 to 1899, the Computing Division was headed by Charles A. Schott, a civil engineer and graduate of the Polytechnic School at Karlsruhe in Germany. Working in the Division at the time were Alexander S. Christie and Myrick Haydon Doolittle. The Computing Division staff size varied, depending on the number of commissions: It had an average of two dozen employees, half permanent and half temporary. During his tenure at the Survey, Kummell authored two articles that appeared in the *Annual Report of the Superintendent of the Coast and Geodetic Survey*, one on a method for computing approximate error, and another on a new method for adjusting a triangulation [**13, 14**].

Research into the theory of errors

Kummell devoted much of his time and effort to studying the probabilistic behavior of random errors. In 1876, he published an article showing that the observational errors in an experiment are normally distributed [**5**].

Kummell reasoned as follows. Suppose that x and x^* are the true and observed values of some quantity, respectively. Then the error

$$\Delta = x^* - x$$

may be seen as consisting of two opposing influences, one tending to make the observation, x^*, greater, and the other smaller than the true value, x. If the positive influence is greater, then $\Delta > 0$, and vice versa. Assume that these influences are made up from $2n$ equal elementary errors, each equal to $+\delta$ or $-\delta$. In the end, we will let $n \to \infty$. Suppose that if $n + k$ positive errors and $n - k$ negative errors occur simultaneously, this results in an error Δ given by

$$\Delta = (n+k)\delta - (n-k)\delta = 2k\delta.$$

Let $d\Delta = 2\delta$. According to Kummell, to produce an error $\Delta + d\Delta$, we must assume that one more $+\delta$ and one less $-\delta$ have occurred simultaneously, to preserve the constant total number of $2n$ errors. Thus,

$$\Delta + d\Delta = (n+k+1)\delta - (n-k-1)\delta = 2(k+1)\delta.$$

Let ϕ_k be the probability of $n + k$ positive errors and $n - k$ negative errors occurring simulataneously. Then the values of ϕ_k are the terms of the expansion of

$$\left(\frac{1}{2} + \frac{1}{2}\right)^{2n},$$

or, in general,

$$\phi_k = \binom{2n}{n+k} \cdot 2^{-2n} = \frac{2n \cdot (2n-1) \cdot \cdots \cdot (n+k+1)}{1 \cdot 2 \cdots 3 \cdots (n-k)} \cdot 2^{-2n}.$$

Kummell treated these probabilities as if they were a continuous function of k. He denoted the general terms ϕ_k and ϕ_{k+1} by ϕ and $\phi + d\phi$, respectively, and divided ϕ_{k+1} by ϕ_k to obtain

$$\frac{\phi + d\phi}{\phi} = \frac{n-k}{n+k+1}.$$

Therefore,

$$\begin{aligned}\frac{d\phi}{\phi} &= -\frac{2k+1}{n+k+1} \\ &= -\frac{2kd\Delta + d\Delta}{nd\Delta + kd\Delta + d\Delta} \\ &= -\frac{2\Delta + d\Delta}{nd\Delta + \Delta + d\Delta} \\ &\approx -\frac{2\Delta}{nd\Delta} \\ &= -\frac{2\Delta d\Delta}{n(d\Delta)^2},\end{aligned}$$

using $kd\Delta = \Delta$. Since $d\Delta = 2\delta$, it follows that $n(d\Delta)^2 = 4n(\delta)^2$ is the double sum of squared elementary errors $\pm\delta$. Putting

$$\epsilon^2 = \frac{n(d\Delta)^2}{2} \tag{3}$$

yields

$$\frac{d\phi}{\phi} = -\frac{\Delta d\Delta}{\epsilon^2}.$$

By integrating the above equation, we obtain

$$\phi = C\exp\left(-\frac{\Delta^2}{2\epsilon^2}\right).$$

Evaluating for $\Delta = 0$ gives $C = \phi_0$. In order to determine ϕ_0, the Wallis product formula was applied:

$$\begin{aligned}\phi_0 &= \binom{2n}{n}2^{-2n} \\ &= \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n} \\ &\approx \frac{1}{\sqrt{n\pi}}.\end{aligned}$$

Therefore, as n tends to infinity, Kummell concluded that

$$\phi_0 = \frac{1}{\sqrt{n\pi}}.$$

From equation (3), we have that

$$n = \frac{2\epsilon^2}{(d\Delta)^2},$$

which leads to

$$\phi = \frac{d\Delta}{\sqrt{2\pi\epsilon^2}}\exp\left(-\frac{\Delta^2}{2\epsilon^2}\right). \tag{4}$$

Modern readers will recognize the normal density function. Under ideal conditions, the mean error of observations, ϵ, can be estimated by

$$\epsilon = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\Delta_i^2},$$

which completely describes the class of observations.

Kummell suggested using the quantity

$$h = \frac{1}{\epsilon\sqrt{2}} \tag{5}$$

as the measure of precision of a system of observations. (The precision of a measurement system, also called reproducibility or repeatability, is the degree to which repeated measurements under unchanged conditions show the same results [**24**].) He also proposed that the precisions of different systems of observations can be compared by means of probable error, r. Kummell's argument was as follows. Suppose that a given system of observations x_i^* is

$$x_i^* = x_i + \Delta_i, \quad i = 1, \ldots, n,$$

where $x_1, \ldots, x_n$ are unknown true values and $\Delta_1, \ldots, \Delta_n$ are errors, respectively. Kummell assumed that all possible errors Δ_i occur exactly in proportion to their probabilities given by (4), which he referred to as "an ideal case". The probable error, r, then is defined as

$$r = \frac{Q_3 - Q_1}{2},$$

where Q_1 and Q_3 are the first and third quartiles of the probability distribution (4). Thus, if Δ is an actual error, then

$$P(\Delta > r) = P(\Delta < r) = \frac{1}{4}, \tag{6}$$

or

$$\frac{1}{\epsilon\sqrt{2\pi}}\int_0^r \exp\left(-\frac{\Delta^2}{2\epsilon^2}\right)d\Delta = \frac{1}{\epsilon\sqrt{2\pi}}\int_r^\infty \exp\left(-\frac{\Delta^2}{2\epsilon^2}\right)d\Delta = \frac{1}{4}.$$

The substitutions

$$\Delta = \epsilon\sqrt{2}t$$

and

$$r = \epsilon\sqrt{2}\rho \tag{7}$$

yield

$$\frac{1}{\sqrt{\pi}}\int_0^\rho \exp\left(-t^2\right)dt = \frac{1}{4\sqrt{2}} = 0.17678.$$

This implies that

$$\rho = 0.47694.$$

Then from (7) it follows that the probable error is

$$r = 0.6745\epsilon.$$

(In modern terms, for normally distributed errors, the median absolute deviation is 0.6745 times the standard deviation.) Notice that h, given by (5), can be now rewritten as

$$h = \frac{\rho}{r}$$

and the probability of an error Δ, given by (4), is

$$\phi = \frac{\rho d\Delta}{r\sqrt{\pi}} \exp\left(-\frac{\rho^2\Delta^2}{r^2}\right).$$

The probable error is highly significant in its relation to a determined quantity. In Kummell's words, "It is then an even wager that the error of determination exceeds the probable error as that it is smaller". Thus, the equation

$$x = a \pm r$$

signifies that the most probable value of the unknown quantity x has been found to be a, with such an uncertainty that the actual error $\Delta = x - a$ has, by equation (6), probability 0.25 of being greater than r and probability 0.25 of being less than $-r$. The probability of any value being equal to x (between x and $x + dx$) is then

$$\phi(x) = \frac{\rho dx}{r\sqrt{\pi}} \exp\left(-\frac{\rho^2(x-a)^2}{r^2}\right).$$

Kummell also showed that if x_1 and x_2 are independent and $x_i = a_i \pm r_i$ for each of $i = 1, 2$, then the most probable value of $y = \alpha_1 x_1 + \alpha_2 x_2$ is

$$y = \alpha_1 a_1 + \alpha_2 a_2 \pm \sqrt{\alpha_1^2 r_1^2 + \alpha_2^2 r_2^2}.$$

He reasoned as follows: The probability of observing y is

$$\phi(y) = \frac{\rho^2 dx_1 dx_2}{r_1 r_2 \pi} \exp\left(-\frac{\rho^2(x_1 - a_1)^2}{r_1^2} - \frac{\rho^2(x_2 - a_2)^2}{r_2^2}\right).$$

After some character-building algebra, the above equation can be rewritten as

$$\phi(y) = \frac{\rho dy}{\pi\sqrt{\alpha_1^2 r_1^2 + \alpha_2^2 r_2^2}} \exp\left(-\frac{\rho^2(y - \alpha_1 a_1 - \alpha_2 a_2)^2}{\alpha_1^2 r_1^2 + \alpha_2^2 r_2^2}\right),$$

and it is maximized when $y = \alpha_1 a_1 + \alpha_2 a_2$. More generally, if

$$y = \sum_{i=1}^{n} \alpha_i x_i, \quad \text{where } x_i = a_i \pm r_i,\ i = 1, \ldots, n$$

then the most probable value is

$$y = \sum_{i=1}^{n} \alpha_i a_i \pm \sqrt{\sum_{i=1}^{n} \alpha_i^2 r_i^2}.$$

This approach cannot be used directly when

$$y = f(x_1, x_2, \ldots, x_n),\ x_i = a_i \pm r_i,\ i = 1, \ldots, n,$$

but the approximate solution can be given, which is closer to perfect, for the smaller the probable errors r_i. Kummell uses Taylor's theorem, neglecting higher-order terms, to rewrite y as a linear combination

$$y = f(a_1, a_2, \ldots, a_n) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(a_1, a_2, \ldots, a_n)(x_i - a_i).$$

The most probable value of y, then, is

$$y = f(a_1, a_2, \ldots, a_n) \pm \sqrt{\sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(a_1, a_2, \ldots, a_n) r_i^2}.$$

Kummell and Merriman

Shortly after Kummell's article appeared in print, Mansfield Merriman, an instructor of civil engineering at the Sheffield Scientific School at Yale, published a bibliography of 408 articles relating to the theory of errors and the method of least squares. Concerning Kummell's 1876 article, Merriman wrote:

> Hagen's proof of 1837 is given abbreviated and improved, and the usual rules for normal equations and probable error are deduced [**17**].

Merriman was referring to the German engineer, Gotthilf Heinrich Ludwig Hagen. In another article published in 1876, Merriman referred to Hagen as "one of the greatest hydraulic engineers of the century" and gave his version of Hagen's derivation, noting that the only two other derivations of the law of errors to appear in English were by P. G. Tait in 1865, who presented it in "greatly modified and less satisfactory form" and Kummell's article, for which he wrote:

> although very abbreviated, and requiring in its readers a previous knowledge of the subject, is very welcome to mathematicians, and it contains one or two modifications of the German method of presentation, which considerably shortens the algebraic work [**18**].

Quick to take offense, Kummell replied:

> The author says, in opening, that a simple and yet perfectly satisfactory proof of the main principle of Least Squares, is still a desideratum. His article being written after he had seen mine, which appeared in the Analyst.... It is to be presumed that he would have improved on my representation of the subject. My paper is very abbreviated, as stated by Mr. Merriman, but is, nevertheless, clear and logical to any careful reader, and gives not a mere glimpse of the theory, but almost everything essential. Mr. Merriman's article contains a number of logical and theoretical blunders, which should not go uncorrected [**6**].

Kummell went on to object to Merriman restricting the problem of least squares to the very special case of the same quantity being measured several times, pointed out several mathematical errors in the article, claimed that some of Hagen's equations were absurd, and asserted that Hagen had not given a complete or consistent proof. He added:

> Mr. Merriman writes that I have given Hagen's proof. Now who would like to be accused of such a thing?

Kummell claimed that his proof was original and his only source was Bartholomew Price's *A Treatise on Infinitesimal Calculus*. He concluded with:

> Mr. Merriman, however, thinks Hagen's proof the best, and he says he gives Hagen's proof in his article. But to do justice to Mr. Merriman, his proof is not near so bad as Hagen's, for the reason that he follows pretty closely Price's presentation, sometimes also adopting, in a disguised form, my own.

The law of errors applied to target shooting

Another topic that attracted Kummell's interest was the theory of errors in target shooting. He was an avid contributor to the *Analyst*, and undoubtedly his interest in target shooting was piqued by a problem posed in the journal in 1878 by Asher B. Evans, principal of Union High School in Lockport, New York:

> If A can plant thirty-six percent of his arrows within a circular target ten inches in diameter at a distance of one hundred yards and B can plant sixty-four percent of his arrows within a circle thirteen and one third inches in diameter at the same distance, prove that B's skill is greater than A's [**3**].

Solutions to the problem were submitted by Kummell; Joseph H. Kerschner, professor of mathematics at Mercersberg College; Pliny Earle Chase, professor of philosophy and logic at Haverford College; and R. J. Adcock, of Monmouth, Illinois. They were followed by an analysis of the problem and the solution by Joel E. Hendricks, editor of the *Analyst*. Several years later, Kummell asserted that Chase's solution was the only correct one and mentioned that he himself was now "investigating the matter from a new point of view [**11**]," by which he probably meant his work on theory of errors tested by rifle target shooting.

In 1884, Kummell spoke on this subject at a meeting of the Mathematical Section of the Philosophical Society of Washington. He based his analysis on Jean Baptiste Joseph Liagre's theory that target shooting is compounded of two distinct independent operations; i.e., sighting (horizontal) and leveling (vertical), each of which is liable to error. Kummell used and extended Sir John Herschel's results regarding distribution of probable errors in target shooting [**12**].

He considered the following problem. Let $(x_1, y_1), \ldots, (x_n, y_n)$ be the coordinates of n shots fired at the target, where x and y measure sighting and leveling, respectively. The probability of hitting a point (x, y) on the target is

$$\frac{dx\,dy}{2\epsilon_x\epsilon_y\pi}\exp\left(-\frac{1}{2}\left(\frac{x^2}{\epsilon_x^2}+\frac{y^2}{\epsilon_y^2}\right)\right),$$

where

$$\epsilon_x = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2} \quad \text{and} \quad \epsilon_y = \sqrt{\frac{1}{n}\sum_{i=1}^{n} y_i^2},$$

respectively. According to Herschel, the shots of equal probability lie on circles. Kummell let r be a radius of such circle and put

$$x_r = \frac{\epsilon}{\epsilon_x}x \quad \text{and} \quad y_r = \frac{\epsilon}{\epsilon_y}y,$$

where

$$\epsilon^2 = \frac{1}{2}\left(\epsilon_x^2 + \epsilon_y^2\right).$$

Then every point on the target corresponds to a point (x_r, y_r) on some equal probability circle $x_r^2 + y_r^2 = r^2$. Notice that the equal probability circle is in the revised coordinate system and is not the same as the circle mentioned in the problem published in the *Analyst*.

The probability of hitting anywhere in the area of the equal probability circle is

$$\frac{r\,dr\,d\alpha}{2\pi\epsilon^2}\exp\left(-\frac{r^2}{2\epsilon^2}\right),$$

where α is the polar angle. The probability of hitting the perimeter of this circle is

$$\frac{r\,dr}{\epsilon^2}\exp\left(-\frac{r^2}{2\epsilon^2}\right). \tag{8}$$

Suppose that n_r is the number of shots falling inside the equal probability circle with radius r. Then

$$\frac{n_r}{n} = \int_0^r \frac{t}{\epsilon^2}\exp\left(-\frac{t^2}{2\epsilon^2}\right)dt = 1 - \exp\left(-\frac{r^2}{2\epsilon^2}\right), \tag{9}$$

where n is total number of shots fired. The value of r that maximizes the probability in (8) is $r = \epsilon$. Then equation (9) implies

$$n_\epsilon = \left(1 - e^{-\frac{1}{2}}\right)n \approx 0.4n;$$

that is, the most probable shot is approximately the distance of the $(0.4n)$th shot from the center.

Now, consider a circle with

$$n_r = \frac{1}{2}n,$$

which Kummell called an *even chance circle*. It follows from (9) that the radius of this circle, ρ, is

$$\rho = \epsilon\sqrt{2\ln 2}.$$

Kummell suggested using the even chance shot, ρ, and the most probable shot, as the measures of marksman's skill, if sighting and leveling are equally good. In addition, he derived a quantity, called the average shot,

$$r_0 = \epsilon\sqrt{\frac{\pi}{2}},$$

and showed that ρ, ϵ, and r_0 are related:

$$\frac{\rho}{\sqrt{2\ln 2}} = \epsilon = r_0\sqrt{\frac{2}{\pi}}.$$

Kummell considered the above formulas complete for practical discussion of target records, provided there is no evidence for a constant vitiating cause. During a shooting match, for example, if the wind is blowing constantly in the same direction, the effect of this might be partially revealed by computing for the whole match the quantity

$$x_0 = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

If the sign of x_0 corresponds to the observed direction of the wind, it might perhaps be proper to refer the shots to the new center, to the right, or left of the true center by x_0. In that case,

$$\epsilon_x = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} x_i^2 - n x_0^2}.$$

In leveling there may be a somewhat constant individual habit of holding a rifle too high or too low, which, however, ought not be eliminated in a fair discussion of a match. This issue could be addressed by computing for each marksman and for a whole team a quantity

$$y_0 = \frac{1}{n} \sum_{i=1}^{n} y_i.$$

In 1882 the average shot, r_0, was introduced by the United States Ordnance Department, under the name "radius of the circle of shots" as a measure of marksman's skill. It replaced the mean absolute deviation of shots.

According to Kummell, the ideal distribution of 45 shots is exhibited in FIGURE 1. Each ring contains 6 shots, leaving 3 shots between the outer ring and infinity. The dotted circle is that of the most probable shot, and the dashed one that of the even chance shot.

TABLE 1 was given in Kummell's paper. It refers to the combined target record of the Irish team at 500 yards' range, in the international shooting match of 1874 at Creedmoor. There were 90 shots fired, 88 of which hit the target. From the actual coordinates of the shots, Kummell calculated the value of the mean square error, ϵ, to be 1.3095 ft. The theoretical number of shots for every given radius is calculated by

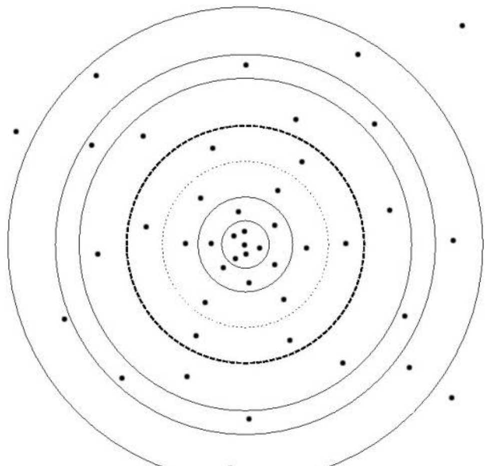

Figure 1 An ideal distribution of 45 shots (reproduced from Kummell's original paper)

TABLE 1: Irish Team at 500 yards: $\epsilon = 1.3095$ ft. The theoretical number of shots for every given radius is calculated by equation (9) (Reproduced from Kummell's original paper).

	Number of shots on circle			Number of shots on ring		
Radii (ft)	Theory	Actual	Discrepancy	Theory	Actual	Discrepancy
0.5	6.3	5	+1.3	6.3	5	+1.3
1.0	22.8	22	+0.8	16.5	17	−0.5
1.5	43.3	47	−3.7	20.5	25	−4.5
2.0	62.0	58	+4.0	18.7	11	+7.7
2.5	75.5	74	+1.5	13.5	16	−2.5
3.0	83.5	83	+0.5	8.0	9	−1.0
3.5	87.5	87+?	+0.5?	4.0	4+?	0.0
4.0	89.2	87+?	?	1.7	?	?
4.5	89.8	88+?	?	0.6	1+?	−0.4?

equation (9). Just prior to his death, Kummell published an article on error analysis concerning discrepant contest data [**15**].

Uncertainty in the independent variables

The method of least squares first appeared in print in an 1805 article by Adrien-Marie Legendre [**16**]. Four years later, assuming that the mean of a set of observations was the most probable value for the object being measured, Gauss derived the law of errors and developed the fundamentals of the least-squares method [**4**]. In least-squares problems, given a data set $\{y_j,\ x_{1j}, \dots, x_{nj}\}_{j=1}^m$, we consider a model

$$y_j = \beta_0 + \beta_1 x_{1j} + \cdots + \beta_n x_{nj} + \epsilon_j, \quad j = 1, \dots, m, \tag{10}$$

where y_j are observed values of a dependent scalar variable Y; $\{x_{1j}, \dots, x_{nj}\}$ are observed values of independent variables $X_1, X_2, \dots X_n$; and ϵ_j are unobservable random errors with mean zero and common standard deviation σ. The unknown coefficients $\beta_0, \beta_1, \dots, \beta_n$ are estimated from the given data using the least-squares method.

In a great many applications, the independent variables cannot be recorded exactly. While employed at the Lake Survey and the U.S. Coast and Geodetic Survey, Kummell often encountered such situations. For instance, comparing standards of length at different temperatures, there are two observed quantities, both of which are measured with a certain amount of error. Assuming that length, y, and temperature, x, are without error, this case can be forced into the ordinary treatment (10), or some other relationship $y = f(x)$ could be considered. However, Kummell treated both dependent and independent variables to be of the form

$$y^* = y + \Delta_y \quad \text{and} \quad x^* = x + \Delta_x,$$

where y^* and x^* are observed values, y and x are unknown true values, Δ_y and Δ_x are random errors, respectively. More generally, he considered the following setup. Let $\{y_j^*,\ x_{1j}^*, \dots, x_{nj}^*\}_{j=1}^m$ be a given data set, where y_j^* are known functions of $x_{1j}^*, \dots, x_{nj}^*$, $j = 1, \dots, m$ [**7**]; that is,

$$\begin{aligned} &f_j * (b_0, b_1, \dots, b_n, y_j^*, x_{1j}^*, \dots, x_{nj}^*) = 0, \\ &x_{ij}^* = x_{ij} + \Delta_{ij}. \end{aligned} \tag{11}$$

The errors Δ_{ij} are assumed to be independent and normally distributed. Kummell assumed that the functions f_j may be different, provided that the same unknown constants $b_0, b_1, \dots, b_n$, either all or in part, occur in them.

Let f_j denote the values in (11), if the true values $x_{1j}, \dots, x_{nj}$ are substituted. Using Taylor's theorem and ignoring higher-order terms leads to the system of equations

$$\begin{aligned}
f_1 + \frac{\partial f_1}{\partial x_{11}}\Delta_{11} + \frac{\partial f_1}{\partial x_{21}}\Delta_{21} + \cdots + \frac{\partial f_1}{\partial x_{n1}}\Delta_{n1} &= 0,\\
f_2 + \frac{\partial f_2}{\partial x_{12}}\Delta_{12} + \frac{\partial f_2}{\partial x_{22}}\Delta_{22} + \cdots + \frac{\partial f_2}{\partial x_{n2}}\Delta_{n2} &= 0,\\
&\vdots\\
f_m + \frac{\partial f_m}{\partial x_{1m}}\Delta_{1m} + \frac{\partial f_m}{\partial x_{2m}}\Delta_{2m} + \cdots + \frac{\partial f_m}{\partial x_{nm}}\Delta_{nm} &= 0.
\end{aligned}$$

To account for different precisions of observed quantities, Kummell proposed to assign weights p_{ij} to x_{ij}. He did not specify how these weights should be chosen, although he mentioned that they should be inversely proportional to squared probable errors of observations. There are weights $P_1, P_2, \dots P_m$ assigned to $f_1, f_2, \dots f_m$ as well, that are defined as follows:

$$\begin{aligned}
\frac{1}{P_1} &= \frac{1}{p_{11}}\left(\frac{\partial f_1}{\partial x_{11}}\right)^2 + \frac{1}{p_{21}}\left(\frac{\partial f_1}{\partial x_{21}}\right)^2 + \cdots + \frac{1}{p_{1m}}\left(\frac{\partial f_1}{\partial x_{n1}}\right)^2\\
\frac{1}{P_2} &= \frac{1}{p_{12}}\left(\frac{\partial f_2}{\partial x_{12}}\right)^2 + \frac{1}{p_{22}}\left(\frac{\partial f_2}{\partial x_{22}}\right)^2 + \cdots + \frac{1}{p_{2m}}\left(\frac{\partial f_2}{\partial x_{n2}}\right)^2\\
&\vdots\\
\frac{1}{P_m} &= \frac{1}{p_{1m}}\left(\frac{\partial f_m}{\partial x_{1m}}\right)^2 + \frac{1}{p_{2m}}\left(\frac{\partial f_m}{\partial x_{2m}}\right)^2 + \cdots + \frac{1}{p_{nm}}\left(\frac{\partial f_m}{\partial x_{nm}}\right)^2
\end{aligned} \tag{12}$$

According to the principle of least squares, we must minimize

$$\sum_{j=1}^{m}\sum_{i=1}^{n} p_{ij}\Delta_{ij}^2. \tag{13}$$

Differentiating the above equation with respect to Δ_{ij}, setting derivatives equal to zero, and using (12), the following expressions for errors are obtained:

$$\Delta_{ij} = -\frac{P_j f_j}{p_{ij}} \cdot \frac{\partial f_j}{\partial x_{ij}}, \quad i = 1, \dots, n,\ j = 1, \dots, m. \tag{14}$$

Substituting (14) into (13) yields

$$\sum_{j=1}^{m} P_j f_j^2,$$

which depends only on observed quantities and unknown coefficients $b_0, \dots, b_n$, whose values are the solutions of

$$\frac{\partial}{\partial b_i}\sum_{j=1}^{m} P_j f_j^2 = 0, \quad i = 0, 1, \dots, n. \tag{15}$$

Kummell noted that only in the simplest case may these conditions be employed to find a direct solution; he used the following example as an illustration. Let

$$f_j^* = ax_j^* - y_j^* + b = 0, \quad j = 1, \ldots, m, \tag{16}$$

where $x_j^* = x_j + \Delta_{x_j}$ and $y_j^* = y_j + \Delta_{y_j}$. Notice that if we let

$$f_j = ax_j - y_j + b, \quad j = 1, \ldots, m,$$

the system (16) can be rewritten as

$$f_j = -a\Delta_{x_j} + \Delta_{y_j}, \quad j = 1, \ldots, m.$$

Let p_j and q_j be the weights of x_j and y_j, respectively. Then by (12) and (15), the values for a and b can be obtained from normal equations

$$\frac{\partial}{\partial a}\left(\sum_{j=1}^{m} \frac{(ax_j + b - y_j)^2}{\frac{a^2}{p_j} + \frac{1}{q_j}}\right) = 0 \quad \text{and} \quad \frac{\partial}{\partial b}\left(\sum_{j=1}^{m} \frac{(ax_j + b - y_j)^2}{\frac{a^2}{p_j} + \frac{1}{q_j}}\right) = 0,$$

where the term to be differentiated represents the weighted sum of squares of lengths of the normals to the required line. Hence, the equations to be solved are

$$\sum_{j=1}^{m} \frac{(ax_j + b - y_j)x_j}{\frac{a^2}{p_j} + \frac{1}{q_j}} - a\sum_{j=1}^{m} \frac{(ax_j - y_j + b)^2}{p_j\left(\frac{a^2}{p_j} + \frac{1}{q_j}\right)^2} = 0,$$

$$\sum_{j=1}^{m} \frac{ax_j + b - y_j}{\frac{a^2}{p_j} + \frac{1}{q_j}} = 0.$$

Alas, unless certain assumptions regarding weights are made, the above system cannot be solved directly. To simplify the problem further, Kummell suggested that weights should have a constant ratio, that is, $q_j = kp_j$, where k is a known constant. He showed that, when $p_j = k = 1$, $j = 1, \ldots, m$, the expression to be minimized is

$$\frac{1}{a^2 + 1}\sum_{j=1}^{m}(ax_j + b - y_j)^2,$$

which yields the solutions

$$a = -\frac{1}{2}M + \sqrt{\frac{1}{4}M^2 + 1} \quad \text{and} \quad b = \frac{1}{m}\left(\sum_{j=1}^{m} y_j - a\sum_{j=1}^{m} x_j\right),$$

where

$$M = \frac{\sum_{j=1}^m x_j^2 - \frac{1}{m}\left(\sum_{j=1}^m x_j\right)^2 - \sum_{j=1}^m y_j^2 + \frac{1}{m}\left(\sum_{j=1}^m y_j\right)^2}{\sum_{j=1}^m x_j y_j - \frac{1}{m}\sum_{j=1}^m x_j \sum_{j=1}^m y_j}.$$

A special case, when weights are equal to one, was also considered by R. J. Adcock [**1**]. His solution for b was identical to Kummell's, but his formula for a was different, due to an algebraic error in elimination of b. (Little is known about R. J. Adcock, 1826–1895. He grew up near Monmouth, Illinois, and taught mathematics at the Kentucky

Military Institute until the Civil War. He then returned to Illinois and became an occasional contributor to the *Analyst* and other early journals.)

In 1890, Merriman, now professor of civil engineering and consultant to the Coast and Geodetic Survey, generalized Kummell's work by minimizing the squares of the vertical distances and deriving formulas for the slope and intercept of the best-fitting line that we recognize today [**20, 21**]. In 1901, Karl Pearson considered a similar least-squares estimation problem. However, he seemed unaware of the previous work by Adcock, Kummell, and Merriman.

Kummell and the Philosophical Society of Washington

The Philosophical Society of Washington was founded in 1871 by Joseph Henry, who served as its president from 1871 to 1878. Other early presidents of the Society were Simon Newcomb, Asaph Hall, and John Wesley Powell. Society meetings were held on alternate Saturdays from October to June. Initially, they took place in the home of Joseph Henry. From 1878 to 1887, general meetings were held in the library of the Army Medical Museum in Ford's Theater on 10th Street. In 1887, they were moved to the Assembly Hall of the Cosmos Club on Florida Avenue, formerly the home of John Wesley Powell. Annual meetings of the Society were held either in late November or early December, often in the law library of Columbian University (now the George Washington University) at 15th and L Streets.

During the period from 1875 to 1897, the average annual membership of the Philosophical Society was about 125. Approximately 29 members and guests attended its general meetings. Persons interested in science who did not reside in the District of Columbia, upon invitation by a member of Society, could be present at all meetings (except for the annual meeting). Several of Kummell's colleagues at the Coast and Geodetic Survey were members of the Society. At first, Kummell attended a few meetings as their guest. On March 25, 1882, he was elected a member.

The aims of the Society were "the promotion of science, the advancement of learning, and the free exchange of views among its members on scientific subjects"; naturally, general meetings were devoted to presentations of scientific work. On December 16, 1876, J. J. Sylvester, who would later found the prestigious *American Journal of Mathematics* at Johns Hopkins University, gave an account of "Some recent investigations into the theory of quaternions." On January 13, 1877, Alexander Graham Bell spoke on his invention of the telephone. A few months later, Marcus Baker talked about the history of Malfatti's problem. Baker worked in the Washington D.C. office of the Coast and Geodetic Survey and, in 1897, served as president of the Philosophical Society. On June 3, 1882, Kummell read a communication on error analysis. An article based on this talk appeared in *Astronomiche Nachrichen* [**10**].

A Mathematical Section of the Philosophical Society was formed on March 29, 1883. Among the thirty-five founders were Doolittle, Hilgard, C. S. Peirce, Hall, Kummell, Hill, and Newcomb (the latter two later served as presidents of the American Mathematical Society). Asaph Hall and civil engineer Henry Farquhar were elected the chair and the secretary, respectively. The objective of the Mathematical Section was the consideration and discussion of papers in pure and applied mathematics. Communications had to be approved by a standing committee of three members, on which Kummell served on several occasions. Membership was open to all members of the Philosophical Society. During the first meeting of the Section, a letter from Marcus Baker was read, which advocated the foundation of a new mathematical journal. At the second meeting on April 11, 1883, it was announced that Moses King, editor of *Science*, had agreed to publish brief reports of the meeting of the Section.

Sixty-eight meetings were held and 120 papers read at the Mathematical Sections from its founding in 1883 until it was disbanded on November 30, 1892. During this time, several attempts were made to establish a national society for the promotion and advancement of mathematics. On January 30, 1884, Kummell read a letter from Artemas Martin, future head of the Library and Archives Division at the Coastal and Geodetic Survey, in which the formation of an American Mathematical Society was proposed. William C. Winlock, of the Naval Observatory, moved the appointment of a special committee with the instruction to report on the advisability of taking steps for the formation of such a society. Hearing no second, Ezekiel B. Elliott, an actuary in the Treasury Department and one of the founders of the Philosophical Society, moved that the matter be postponed. Certainly, an opportunity was lost, but not for long. Four years later, on November 24, 1888, the New York Mathematical Society, precursor to the American Mathematical Society, was founded. The American Mathematical Society was formed in 1894. It was joined by many members of the Mathematical Section of the Philosophical Society, and several of them served as its officers.

Conclusion

In a time of increased awareness of professionalism in the mathematical and statistical sciences, C.H. Kummell worked diligently as a statistician for the Computing Division of the U.S. Coast and Geodetic Survey. His numerous publications covered a wide range of mathematical and statistical topics, but Kummell's major accomplishments were in the theory of least squares and error analysis. He was an outstanding self-taught mathematician and musician, who excelled in both areas.

Besides his work on error analysis, Kummell published thirty articles on such topics as Cauchy's theory of residues, differential geometry, abstract algebra, elliptic functions, geometry, astronomy, and physics. Notable at the time were his articles justifying and further improving a method of evaluating square roots by Clinton Avery Roberts. He contributed problems and solutions, many of which dealt with approximations and error analysis, to the *Analyst*, *Mathematical Magazine* [not this MAGAZINE], *Mathematical Visitor*, and the English pedagogical journal, the *Educational Times and Journal of the College of Preceptors*.

Kummell died on April 17, 1897, of pneumonia. A Computing Division obituary note remarked that he was

> an able mathematician, who at various times contributed valuable mathematical pages and solutions to intricate problems to the mathematical and astronomical journals and to the annual reports to the Coast and Geodetic Survey. His knowledge and experience and his amiable disposition made him a valued member of the computing force of the office, and one whose loss is much and sincerely regretted [**22**].

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REFERENCES

1. R. J. Adcock, Note on the method of least squares, *Analyst* **4** (1877) 183–184; **5** (1878) 21–22, 53–54.
2. M. Baker, "Charles Hugh Kummell, 1836–1897," *Bulletin of The Philosophical Society of Washington* **13** (1900) 404–405.

3. A. B. Evans, Problem 239, *Analyst* **5** (1878) 191.
4. C. F. Gauss, *Theoria Motus Corporum Coelestium in sectionibus conicis solem ambientium,* Hamburg, 1809.
5. C. H. Kummell, New Investigation of the Law of Errors of Observations, *Analyst* **3** (1876) 132–140, 165–171.
6. ———, Remarks on Mr. Merriman's (Ph.D.) article, entitled: "An elementary discussion of the principle of least squares," *Journal of the Franklin Institute* **104** (1877) 173–187.
7. ———, Revision of proof of the formula for the error of observation, *Analyst* **6** (1879) 80–81.
8. ———, Reduction of observation equations which contain more than one observed quantity, *Analyst* **4** (1879) 97–105.
9. ———, Proof of some remarkable relations in the method of least squares, *Analyst* **7** (1880) 84–88.
10. ———, On the composition of errors from single causes of error, *Astronomische Nachrichten* **103** (1882) 117.
11. ———, Reconsideration of the solution to problem 239, *Analyst* **10** (1883) 121–123.
12. ———, The theory of errors practically tested by target-shooting, *Bulletin of the Philosophical Society of Washington* **6** (1884) 138–148.
13. ———, On an approximate method of computing probable error. *Report of the Superintendent of the U.S. Coast and Geodetic Survey showing the Progress of the Work during the fiscal year ending with June, 1890.* U.S. Government Printing Office, Washington, DC, 1891, 685–687.
14. ———, On the direct synthetic method of adjusting a triangulation. *Report of the Superintendent of the U.S. Coast and Geodetic Survey Showing the Progress of the Work for the Fiscal Year Ending June 1892*, U.S. Government Printing Office, Washington, DC, 1893, Appendix 12, 535–552.
15. ———, Discussion of merit contests in college examinations by the method of least squares, *American Mathematical Monthly* **4** (1897) 129–132.
16. A. M. Legendre, *Nouvelles méthodes pour la d'etermination des orbites des comètes*, Paris, 1805.
17. M. A. Merriman, A list of writings related to the method of least squares, with historical and critical notes, *Transactions of the Connecticut Academy of Arts and Sciences* **4** (1877) 151–232.
18. ———, An elementary discussion of the principle of least squares, *Journal of the Franklin Institute* **104** (1877) 270–274.
19. ———, *Elements of Least Squares*, Macmillan, London, 1877.
20. ———, A problem in Least Squares, *Bulletin of the American Mathematical Society* **1** (1891) 39–42. http://dx.doi.org/10.1090/S0002-9904-1891-00021-8
21. ———, The determination, by the method of least squares, of the relation between two variables, connected by the equation $y = ax + b$, both variables being liable to errors of observation, *Report of the Superintendent of the U.S. Coast and Geodetic Survey showing the Progress of the Work during the fiscal year ending with June, 1890*, U.S. Government Printing Office, Washington, DC, 687–690.
22. C. A. Schott, Obituary notice. *Report of the Superintendent of the Coast and Geodetic Survey Showing the Progress of the Work During the Fiscal Year Ending with June 1897*, U.S. Government Printing Office, Washington, DC, 1898, p. 57.
23. S. M. Stigler, Mathematical Statistics in the Early States, *Annals of Statistics* **6** (1978) 239–265.
24. J. R. Taylor, *An Introduction to Error Analysis: The Study of Uncertainties in Physical Measurements*, 2nd ed., University Science Books, Mill Valley, CA, 1996, pp. 128–129.
25. R. C. Woodward, *Smithsonian Geographical Tables* vi, Smithsonian Institution, Washington, DC, 1894.

Summary European statistical research flourished in the years 1770 to 1850 whereas statistical research in the United States did not develop fully until the latter half of the nineteenth century. The establishment of the United States Coast Survey, the Lake Survey, and the Nautical Almanac in 1807, 1841, and 1849, respectively, encouraged the rapid advancement of interest in statistics and the development of statistical methods. It was pioneered by Robert Adrain, Benjamin Peirce, his son Charles Sanders Peirce, Simon Newcomb, and Erastus Lyman De Forest, whose work is well researched. In this article we focus on life and the statistical accomplishments of Charles Hugo Kummell, a statistician for the Lake Survey and the U.S. Coast and Geodetic Survey, and an active member of the Philosophical Society of Washington. We describe his research into laws of errors of observations and his contributions to the development of the least-squares method.

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Quartic Coincidences and the Singular Value Decomposition

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A flurry of articles by Kalman [**4, 5, 6**] and by Minda and Phelps [**11**] have reinvigorated interest in a century-old theorem.

BÔCHER-GRACE THEOREM. *Suppose that $\mathcal{T}$ is a triangle with non-collinear vertices z_1, z_2, and z_3 in the complex plane. There is a unique ellipse $\mathcal{E}$ inscribed in $\mathcal{T}$ that passes through the midpoints of its sides. The ellipse $\mathcal{E}$ has the largest area among all ellipses inscribed in $\mathcal{T}$, and its major axis is along the orthogonal best fit line to the three vertices. Further, if*

$$p(z) = (z - z_1)(z - z_2)(z - z_3)$$

is the monic polynomial with roots at the vertices, then the critical points of $p(z)$ are the foci of $\mathcal{E}$.

The theorem was proved by Siebeck [**12**], Bôcher [**1**], and Grace [**7**]. Kalman calls it "Marden's Theorem" based on its presentation in Marden's book [**10**]. We are following Marden himself in crediting Bôcher and Grace.

In this article, we will show under what conditions this remarkable theorem extends to quadrilaterals and fourth-degree polynomials. We use only methods from elementary linear algebra, most notably the Singular Value Decomposition Theorem.

From cubics to quartics

The Bôcher-Grace Theorem is really a theorem about geometry, combined with an algebraic coincidence involving the complex plane. In any triangle $\mathcal{T}$ there is a two-parameter family of inscribed ellipses. The largest among these in terms of enclosed area is called the Steiner ellipse [**13**], and it passes through the midpoints of the sides of $\mathcal{T}$. In addition, the major axis of symmetry of the Steiner ellipse is along a line that provides the orthogonal best fit to the vertices of the triangle $\mathcal{T}$, meaning that it minimizes the sum of the squared distances from the vertices to the line, each distance being measured orthogonally to the line as in FIGURE 1 [**11**].

The properties of the Steiner ellipse are all very interesting, but have no apparent connection to the critical points of complex polynomials. Now, enter the Bôcher-Grace Theorem. If p is a complex cubic polynomial vanishing at the vertices of $\mathcal{T}$, then its critical points coincide with the foci of the Steiner ellipse for $\mathcal{T}$!

Our main result extends these results to parallelograms and quartics.

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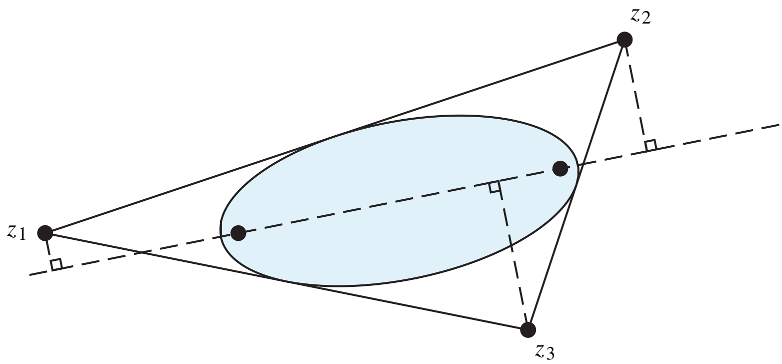

Figure 1 The critical points of $p(z) = (z - z_1)(z - z_2)(z - z_3)$ are the foci of the Steiner ellipse.

BÔCHER-GRACE THEOREM FOR QUARTICS. *Suppose that $\mathcal{P}$ is a parallelogram with vertices z_1, z_2, z_3, and z_4 in the complex plane. There exists a unique ellipse $\mathcal{E}$ inscribed in $\mathcal{P}$ that passes through the midpoints of all of its sides. The ellipse $\mathcal{E}$ has the largest area among all ellipses inscribed in $\mathcal{P}$, and its major axis is along the orthogonal best fit line to the four vertices. Further, if*

$$p(z) = (z - z_1)(z - z_2)(z - z_3)(z - z_4)$$

is the monic polynomial with roots at the vertices, then the critical points of $p(z)$ are the foci and the center of $\mathcal{P}$.

If $\mathcal{P}$ is a convex quadrilateral other than a parallelogram, then it has no inscribed ellipse that passes through the midpoints of all of its sides.

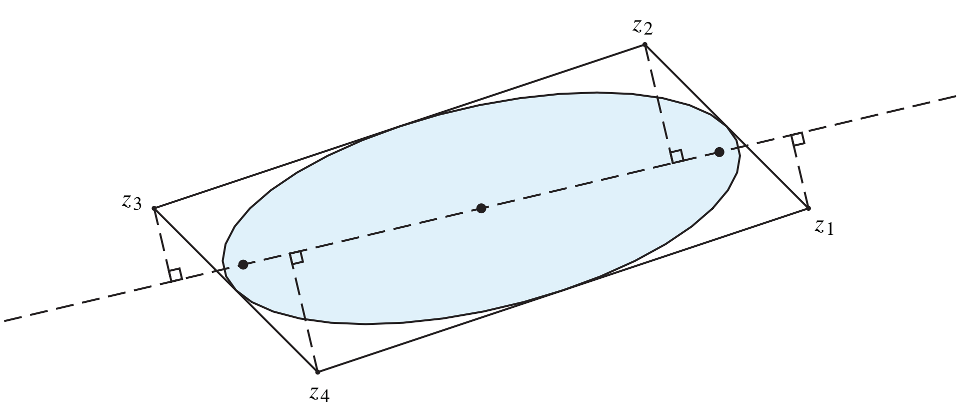

Figure 2 The critical points of $p(z) = (z - z_1)(z - z_2)(z - z_3)(z - z_4)$ are the foci and center of the Steiner ellipse.

The fact that the critical points of a quartic vanishing in the vertices of a parallelogram lie on the orthogonal best fit line to those vertices was known to Coolidge [**3**], and some recent proofs are by Keady [**8**] and Minda and Phelps [**11**]. Our proof of this fact is novel in that it makes use of the Singular Value Decomposition Theorem. The fact that the ellipse interpolating the midpoints of the parallelogram is the inscribed ellipse with the largest area, what we shall call the Steiner ellipse of the parallelogram, is known. Our proof follows by exhibiting implicit equations for the one-parameter family of inscribed ellipses. What is new is the connection between quadrilaterals that have ellipses interpolating the midpoints of their sides and the critical points of quartics that have roots at their vertices. We can view this as a special case of Theorem 4.2 in Marden [**10**].

The Singular Value Decomposition Theorem

The Singular Value Decomposition (SVD) Theorem guarantees that for each $n \times n$ matrix A with entries in $\mathbb{R}$, there exist orthogonal $n \times n$ matrices U and V and a nonnegative diagonal $n \times n$ matrix S that satisfy

$$AV = US.$$

Introductions to this theorem are found in the books by Lay [**9**] and Strang [**14**]. The matrices U, V, and S are not necessarily unique in all cases; convention dictates that the diagonal entries of S are arranged in decreasing order. Since the inverse of an orthogonal matrix is its transpose, the SVD Theorem implies a factorization of A:

$$A = USV^T.$$

The usefulness of this decomposition becomes apparent upon labeling the columns of U and V, and the diagonal entries of S:

$$A \begin{bmatrix} v_1 & v_2 & \cdots & \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \end{bmatrix}.$$

As the column vectors of an orthogonal matrix, the vectors $v_1, \ldots, v_n$ are all unit vectors and are mutually orthogonal. The same is true of the vectors $u_1, \ldots, u_n$. Applying A to each of the columns of V yields a relationship among these unit vectors and the values σ_k,

$$Av_k = \sigma_k u_k, \tag{1}$$

for each k. Equation (1) resembles the relationship between eigenvectors and eigenvalues, and that inspires the terminology employed for the vectors and values, replacing the prefix *eigen* with the word *singular*. That is, the columns of U and V are referred to as *singular vectors* of A, *left* and *right*, respectively, while the diagonal entries of S are its *singular values*.

The SVD Theorem has a marvelous geometrical interpretation: The image of the unit sphere under matrix multiplication is an ellipsoid. (It is non-degenerate only if all of the singular values are positive.) To see this, we express the vectors of the unit sphere $\mathcal{S}$ thus: $\{V\alpha : \|\alpha\| = 1, \alpha \in \mathbb{R}^n\}$. To determine the image of $\mathcal{S}$ under the mapping A, we look at the sequence of images $V^T\mathcal{S}$, $SV^T\mathcal{S}$, and $USV^T\mathcal{S}$. Because multiplication by an orthogonal matrix preserves lengths and angles, application of the factor V^T to the vectors of $\mathcal{S}$ yields a reorientation of $\mathcal{S}$ (see FIGURE 3). Multiplication of the vectors of the unit sphere by a nonnegative diagonal matrix such as S expands or contracts those Euclidean directions for which the diagonal entries are positive, and collapses to zero those that are zero, creating an ellipsoid. Finally, multiplication of this ellipsoid by the orthogonal matrix U preserves the dimensions of the ellipsoid, simply reorienting the ellipsoid into a new position about the origin.

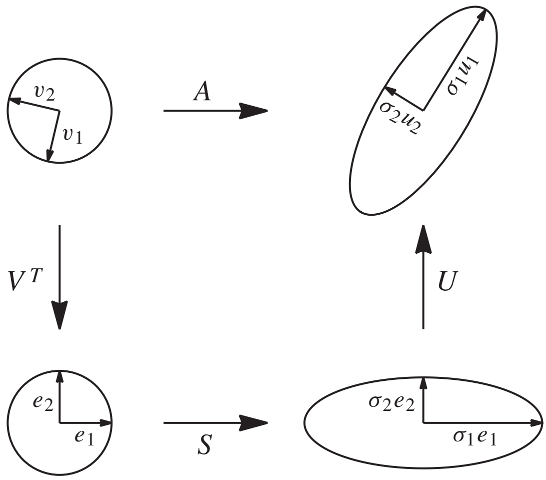

Figure 3 Matrices map circles to ellipses

From the discussion above, it should be evident that the singular vectors u_k are parallel to the axes of symmetry of the ellipsoid. Because the singular vectors are unit vectors, it follows that the singular values themselves correspond to semi-axis lengths of the ellipsoid.

In this article, we shall make use of the SVD Theorem directly or indirectly several times. We first illustrate the SVD Theorem by computing the decomposition for a class of 2-by-2 matrices.

SVD of triangular matrix

Consider the triangular matrix A and its inverse:

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \qquad A^{-1} = \frac{1}{ac}\begin{bmatrix} c & -b \\ 0 & a \end{bmatrix}, \tag{2}$$

where $a, b, c > 0$. We express the SVD Theorem $A\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} S$ in terms of trigonometric functions:

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}.$$

In this formulation, v_1 makes an angle θ with the x-axis; u_1 makes an angle φ with the x-axis. The angles θ and ϕ are chosen so that $\sigma_1 > \sigma_2$.

Given that u_1 and u_2 are orthonormal, (1) implies that

$$(Av_1)\cdot(Av_2) = (\sigma_1 u_1)\cdot(\sigma_2 u_2) = \sigma_1\sigma_2(u_1\cdot u_2) = 0;$$
$$(Av_k)\cdot(Av_k) = (\sigma_k u_k)\cdot(\sigma_k u_k) = \sigma_k^2(u_k\cdot u_k) = \sigma_k^2.$$

Thus, we can determine θ by solving $Av_1 \cdot Av_2 = 0$, and we can determine the singular values σ_k by computing $Av_k \cdot Av_k$. Using the double-angle identities $\sin 2\theta =$

$2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$, we can express

$$Av_1 \cdot Av_2 = \frac{b^2+c^2-a^2}{2}\sin 2\theta + ab\cos 2\theta, \tag{3}$$

$$Av_k \cdot Av_k = \frac{a^2+b^2+c^2}{2} \pm \frac{a^2-b^2-c^2}{2}\cos 2\theta \pm ab\sin 2\theta, \tag{4}$$

where in equation (4) the positive signs correspond to the case $k = 1$, and the negative signs to $k = 2$. Equation (3) and the requirement that $Av_1 \cdot Av_2 = 0$ imply that

$$\tan 2\theta = \frac{2ab}{a^2-b^2-c^2}. \tag{5}$$

Since $a, b, c > 0$, the magnitude of Av_1 is largest when the components of $v_1 = (\cos\theta, \sin\theta)$ are positive. For this reason, among the four values of θ satisfying equation (5), we select the one that lies in the interval $[0, \pi/2]$. This choice completely determines $V = \left[v_1\ v_2\right]$.

To determine the singular values σ_k, we incorporate the value for θ into (4) by identifying the numerator in (5) with $\sin 2\theta$ and the denominator with $\cos 2\theta$. Then

$$\sin 2\theta = 2ab/\sqrt{(a^2-b^2-c^2)^2+(2ab)^2},$$

and

$$\cos 2\theta = (a^2-b^2-c^2)/\sqrt{(a^2-b^2-c^2)^2+(2ab)^2}.$$

Substituting these expressions for $\sin 2\theta$ and $\cos 2\theta$ into the equation (4), we get

$$\sigma_k^2 = \frac{1}{2}\left(a^2+b^2+c^2 \pm \sqrt{(a^2+b^2-c^2)^2+(2bc)^2}\right), \tag{6}$$

where once again $k = 1$ corresponds to the positive sign, and $k = 2$ to the negative sign.

So far, we have defined the matrices V and S in the decomposition $AV = US$. To determine U, we use the companion decomposition for the inverse of A, which is given by $A^{-1}U = VS^{-1}$. The angle φ can be determined using the method above, but by insisting that $A^{-1}u_1 \cdot A^{-1}u_2 = 0$. Alternatively, since the formula for θ in (5) was derived for an arbitrary 2-by-2 upper triangular matrix, it can be used to define φ for the upper triangular matrix A^{-1} from equation (2). With either approach, we conclude that

$$\tan 2\varphi = \frac{2bc}{a^2+b^2-c^2}. \tag{7}$$

We close this example by examining the geometrical interpretation of the SVD Theorem. The triangular matrix A in (2) maps a circle of radius R centered at the origin to an ellipse. We now know that the angle φ in (7) corresponds to the major axis of symmetry of the image ellipse. The major and minor semi-axis lengths $\sigma_1 R$ and $\sigma_2 R$ can be determined from (6). From the semi-axis lengths, we can compute the distance from the foci to the center of the ellipse:

$$\sqrt{\sigma_1^2-\sigma_2^2}\,R = \left((a^2+b^2-c^2)^2+(2bc)^2\right)^{1/4}R. \tag{8}$$

Parametrization of an ellipse

Let $\mathcal{E}$ denote an ellipse centered at the origin. Let u_1 and u_2 correspond to unit vectors parallel to its major and minor axes of symmetry, and the constants $\sigma_1 \geq \sigma_2 > 0$ to its major and minor semi-axis lengths. Then $\mathcal{E}$ can be parametrized by

$$\sigma_1 \begin{bmatrix} u_1 \end{bmatrix} \cos t + \sigma_2 \begin{bmatrix} u_2 \end{bmatrix} \sin t = \begin{bmatrix} u_1 \; u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} =: USu(t).$$

This formula *parametrizes* the ellipse by establishing a 1-to-1 relationship between points on the ellipse and values of t in $[0, 2\pi)$. Since both U and S are invertible, $\mathcal{E}$ can be mapped to the unit circle by applying the product $S^{-1}U^T$ to $\mathcal{E}$. If $\mathcal{E}$ were inscribed in a quadrilateral $\mathcal{Q}$ and interpolating its midpoints—that is, passing through the midpoints of the sides—then $S^{-1}U^T\mathcal{Q}$ would correspond to a quadrilateral circumscribing the unit circle and whose midpoints lie on the circle. That is, the pre-image of $\mathcal{Q}$ would need to be a square, and $\mathcal{Q}$ would need to be a parallelogram. Thus *no ellipse can be inscribed in a convex quadrilateral interpolating its midpoints unless the quadrilateral is a parallelogram.*

Ellipses inscribed in a diamond

There is a one-parameter family of ellipses that can be inscribed in the diamond $\mathcal{D}$ with vertices $(\pm 1, 0)$ and $(0, \pm 1)$. If, for fixed $0 < t < 1$, we insist that the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{9}$$

pass through the point $(t, 1-t)$ and be tangent to the line $x + y = 1$ at that point, then we have two constraints that must be satisfied by a^2 and b^2:

$$\frac{t^2}{a^2} + \frac{(1-t)^2}{b^2} = 1 \quad \text{and} \quad \frac{t}{1-t}\frac{b^2}{a^2} = 1.$$

Together these imply that the desired ellipse takes the form

$$\frac{x^2}{t} + \frac{y^2}{1-t} = 1. \tag{10}$$

It is not difficult to verify that each ellipse is inscribed in $\mathcal{D}$, since it is tangent to the segments of $\mathcal{D}$ at the four points $\big(\pm t, \pm(1-t)\big)$.

Ellipses inscribed in a canonical parallelogram

We are interested in ellipses inscribed within an arbitrary parallelogram. Every parallelogram can be translated so that the intersection of its diagonals lies at the origin. Furthermore, every parallelogram centered at the origin can be rotated so that one of its diagonals lies on the x-axis and one of its vertices lies in the first quadrant. Such a parallelogram, with vertices $\pm(a, 0)$ and $\pm(b, c)$, for $a, b, c > 0$, shall be referred to as a *canonical parallelogram*.

The image of the diamond $\mathcal{D}$ under the triangular matrix A defined in (2) is a canonical parallelogram $\mathcal{P}$. The image of the family of inscribed ellipses in $\mathcal{D}$ defined in (10)

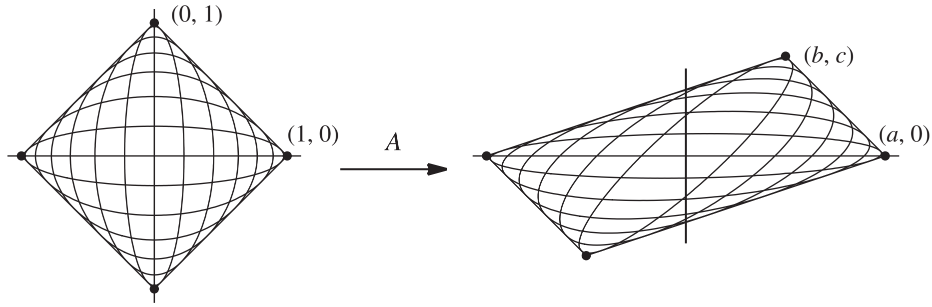

Figure 4 Inscribed ellipses in $\mathcal{D}$ and in $\mathcal{P}$

under multiplication by A becomes

$$\frac{(cx-by)^2}{t(ac)^2}+\frac{y^2}{(1-t)c^2}=1, \tag{11}$$

a family of ellipses inscribed in the parallelogram $\mathcal{P}$. This equation is determined by considering the pre-image of the ellipses inscribed in $\mathcal{P}$. If the ordered pair (x, y) lies on an ellipse in $\mathcal{P}$, then the inverse image of this point under A^{-1} is $(cx-by, ay)/ac$ and must lie on an ellipse in $\mathcal{D}$.

Among the ellipses inscribed in $\mathcal{P}$, the one that has the largest area, the Steiner ellipse, corresponds to $t=\frac{1}{2}$. To see this, recall that the area bounded by the general ellipse (9) is πab; hence, the area of each member of the family of ellipses (10) must be $\pi\sqrt{t(1-t)}$. Since the area of a geometric region under a matrix mapping A is scaled by the determinant of that matrix, it follows that the area bounded by each of the ellipses defined by (11) is $\pi\sqrt{t(1-t)}ac$.

Of course the pre-image of the Steiner ellipse is the circle of radius $R=1/\sqrt{2}$ inscribed in the diamond $\mathcal{D}$. Since this circle interpolates the midpoints of $\mathcal{D}$, and matrix multiplication preserves proportions, it follows that *the Steiner ellipse interpolates the midpoints of $\mathcal{P}$.*

Complex quartic polynomials

Having established the asserted properties of Steiner ellipses for parallelograms, we now complete the proof of the main result by showing that the critical points of a complex polynomial vanishing at the vertices of a parallelogram in the complex plane coincide with the center and foci of the parallelogram's Steiner ellipse.

Take p to be the monic polynomial that vanishes at $\pm a$ and at $\pm(b+ci)$, the vertices of a canonical parallelogram $\mathcal{P}$. Then

$$p(z)=\left(z^2-a^2\right)\left(z^2-(b+ci)^2\right),$$

while its derivative, a cubic polynomial, is

$$p'(z)=2z\Big(2z^2-\left(a^2+(b+ci)^2\right)\Big).$$

Since the roots of the quadratic factor of $p'(z)$ differ only in sign, let these two roots be denoted by $re^{i\varphi}$ and $-re^{i\varphi}$. Then

$$r^2e^{i2\varphi} = \frac{1}{2}\big(a^2 + (b+ic)^2\big) = \frac{1}{2}\big(a^2 + b^2 - c^2 + i2bc\big),$$

from which it follows that

$$r^4 = \frac{1}{4}\big((a^2+b^2-c^2)^2 + (2bc)^2\big) \quad \text{and} \quad \tan 2\varphi = \frac{2bc}{a^2+b^2-c^2}. \tag{12}$$

A direct comparison of the defining equations for r and φ in equation (12) with the focal distance defined in (8) for $R = 1/\sqrt{2}$, and with θ defined in equation (7), demonstrates that $\varphi = \theta$. Thus, *the roots of* $p'(z)$ *coincide with the foci and center of the Steiner ellipse of* $\mathcal{P}$.

Orthogonal best fit line

We now show that the *orthogonal best fit line* (OBFL) to the vertices of a parallelogram coincides with the major axis of symmetry of its Steiner ellipse. The OBFL to a collection of points $\{(x_k, y_k) : k = 1, \ldots, m\}$ with centroid (x_0, y_0) is a line of the form

$$\beta_0(x - x_0) + \beta_1(y - y_0) = 0,$$

where the values of β_0 and β_1 are chosen to minimize the sum of the squares of the *perpendicular distances* of the points to the line:

$$SS_\perp = \frac{1}{\beta_0^2 + \beta_1^2} \sum_{k=1}^{m} \big(\beta_0(x_k - x_0) + \beta_1(y_k - y_0)\big)^2.$$

The perpendicular distance from the point (x_k, y_k) to the line $\beta_0(x - x_0) + \beta_1(y - y_0) = 0$ is the scalar part of the projection of the vector $\langle x_k - x_0, y_k - y_0\rangle$ onto the line's normal vector $\langle \beta_0, \beta_1\rangle$.

The selection of parameters β_0 and β_1 to minimize $SS_\perp$ was addressed by J. L. Coolidge in 1913 [**3**]. There, he made the substitution

$$\cos\alpha = \frac{\beta_0}{\sqrt{\beta_0^2 + \beta_1^2}} \quad \text{and} \quad \sin\alpha = \frac{\beta_1}{\sqrt{\beta_0^2 + \beta_1^2}},$$

and took advantage of trigonometric identities to determine β_0 and β_1. Here we take a different approach, using the SVD Theorem.

Define the matrix $M \in \mathbb{R}^{m\times 2}$ and an arbitrary unit vector u by

$$M = \begin{bmatrix} x_1 - x_0 & y_1 - y_0 \\ \vdots & \vdots \\ x_m - x_0 & y_m - y_0 \end{bmatrix} \quad \text{and} \quad u = \frac{1}{\sqrt{\beta_0^2 + \beta_1^2}} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}. \tag{13}$$

A straightforward calculation shows that $SS_\perp = \|Mu\|^2$. Thus, minimizing $SS_\perp$ over all pairs β_0 and β_1 is equivalent to determining the smallest value of $\|Mu\|$ over all possible unit vectors u.

Thin SVD

The centroid of the points $\pm(a, 0)$ and $\pm(b, c)$ is the origin, and so the matrix M in (13) can be written as

$$M = \begin{bmatrix} A^T \\ -A^T \end{bmatrix} = \begin{bmatrix} VSU^T \\ -VSU^T \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}V \\ -\frac{1}{\sqrt{2}}V \end{bmatrix} (\sqrt{2}S)U^T = \widehat{V}\,\widehat{S}\,U^T,$$

where A is the triangular matrix in (2), and we make use of the factorization $A^T = VSU^T$. Since U is an orthogonal matrix, the factorization $M = \widehat{V}\widehat{S}U^T$ leads to the *thin* SVD of M,

$$MU = \widehat{V}\,\widehat{S}.$$

The thin SVD captures the essence of the geometrical action of a matrix, focusing exclusively on those vectors of the unit sphere that are mapped directly to the image ellipsoid, ignoring those in the null space that are mapped to $\mathbf{0}$.

Since the matrix M maps the unit circle to an ellipse, the magnitudes of the image vectors $\|Mu\|$ can be no larger than $\sqrt{2}\sigma_1$ and no smaller than $\sqrt{2}\sigma_2$. Thus, the minimum of $\|Mu\|$ is $\sqrt{2}\sigma_2$, and it is attained when $u = \pm u_2$, u_2 being the second orthonormal vector in matrix U. The components of u_2 are $(-\sin\varphi, \cos\varphi)$ for the angle φ defined in (7). It is these values that serve as the coefficients for the OBFL, leading to an implicit equation

$$(-\sin\varphi)x + (\cos\varphi)y = 0,$$

a line that is parallel to the direction u_1. Thus *the major axis of the Steiner ellipse for $\mathcal{P}$ is the orthogonal best fit line to the vertices of $\mathcal{P}$.*

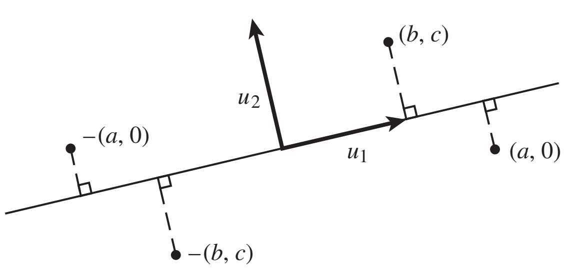

Figure 5 OBFL to parallelogram vertices

Concluding remarks

It is natural to ask what happens in the cases beyond triangles and quadrilaterals. As it happens, there exists an inscribed ellipse interpolating the midpoints of a convex polygon if and only if the polygon is an affine image of a regular polygon. In this case, the critical points of a polynomial vanishing at the vertices of the polygon are collinear, and the two critical points farthest apart coincide with the foci of the Steiner ellipse. The authors make use of the Chebyshev polynomials to demonstrate this fact in [**2**].

Acknowledgment The authors wish to acknowledge the extremely constructive comments of the referee that helped us better integrate the SVD into our exposition.

REFERENCES

1. M. Bôcher, Some propositions concerning the geometric representation of imaginaries, *Ann. Math.* **7v** (1892) 70–76.
2. J. Clifford and M. Lachance, A generalization of the Bocher-Grace Theorem, to appear in *Rocky Mountain J. Math.*
3. J. L. Coolidge, Two geometrical applications of the method of least squares, *Amer. Math. Monthly* **20** (1913) 187–190. http://dx.doi.org/10.2307/2973072
4. D. Kalman, The most marvelous theorem in mathematics, *Journal of Online Mathematics and Applications*, 2008.
5. D. Kalman, An elementary proof of Marden's Theorem, *Amer. Math. Monthly* **115** (2008) 330–338.
6. D. Kalman, The most marvelous theorem in mathematics, *Math Horizons,* 2008.
7. J. H. Grace, The zeros of a polynomial, *Proc. Cambridge Philos. Soc.* **11** (1902) 352–357.
8. G. Keady, Lines of best fit for the zeros and for the critical points of a polynomial, *Amer. Math. Monthly* **118** (2011) 262–264.
9. D. C. Lay, *Linear Algebra and Its Applications*, 3rd ed., Addison Wesley, Reading, MA, 2003.
10. M. Marden, *Geometry of Polynomials*, 2nd ed., Mathematical Surveys Number 3, American Mathematical Society, Providence, RI, 1966.
11. D. Minda and S. Phelps, Triangles, ellipses, and cubic polynomials, *Amer. Math. Monthly* **115** (2008) 679–689.
12. J. Siebeck, Ueber eine neue analytsche Behandlungweise der Brennpunkte, *J. Reine Angew. Math.* **64** (1864) 175.
13. J. Steiner, *Gesammelte Werke*, vol. 2, Prussian Academy of Sciences, Berlin, 1881–1882.
14. G. Strang, *Linear Algebra and Its Applications*, 3rd ed., Harcourt Brace Jovanovich, San Diego, CA, 1988.

Summary The singular value decomposition is a workhorse in many areas of applied mathematics and the insights it gives to linear transformations is beautiful. Using the geometry given by the SVD, we prove that the critical points of a quartic polynomial whose zeros are the vertices of a parallelogram are the foci and center of an inscribed ellipse passing through the midpoints of the sides of the parallelogram.

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Proof Without Words: The Formulas of Hutton and Strassnitzky

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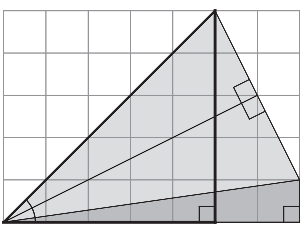
Hutton's formula: $$\frac{\pi}{4} = 2\arctan\frac{1}{3} + \arctan\frac{1}{7} \tag{1}$$

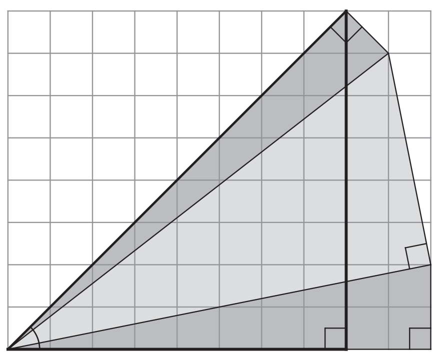
Strassnitzky's formula: $$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{5} + \arctan\frac{1}{8} \tag{2}$$

Proof:

(1)



(2)



NOTE: Charles Hutton published (1) in 1776, and in 1789 Georg von Vega used it with Gregory's arctangent series to compute π to 143 decimal places, of which the first 126 were correct. L. K. Schulz von Strassnitzky provided (2) to Zacharias Dahse in 1844, who then used it to compute π correct to 200 decimal places [**1**].

REFERENCE

1. L. Berggren, J. Borwein, and P. Borwein, *Pi: A Source Book,* Springer, New York, 1997. http://dx.doi.org/10.1007/978-1-4757-2736-4

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NOTES

On a Geometrical Formula Involving Medians and Bimedians

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The following proposition concerning a triangle is well known.

PROPOSITION 1. *For any triangle, the sum of the squares of its three medians is equal to three fourths of the sum of the squares of its sides.*

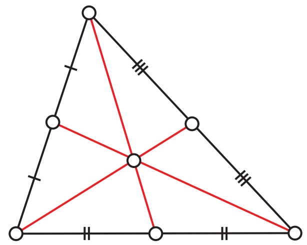

Figure 1 The medians of a triangle

A bit less well known is that there are analogous formulas for the tetrahedron. Indeed, defining a median as the segment joining a vertex to the barycenter of the opposite face (FIGURE 2), we have the following [**1**, page 57].

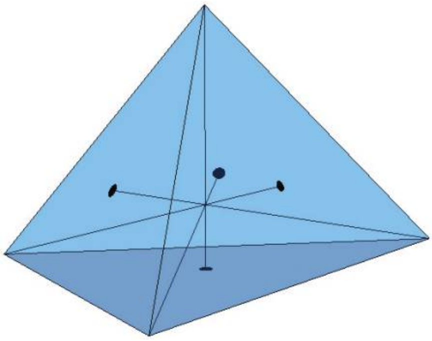

Figure 2 The medians of a tetrahedron

PROPOSITION 2. *For any tetrahedron, the sum of the squares of its four medians is equal to four ninths of the sum of the squares of its edges.*

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A quite different analogue is obtained if we consider bimedians instead of medians. Recalling that a bimedian is the segment joining the midpoints of two opposite edges of the tetrahedron (FIGURE 3), we have the following [**1**, p. 56] [**2**].

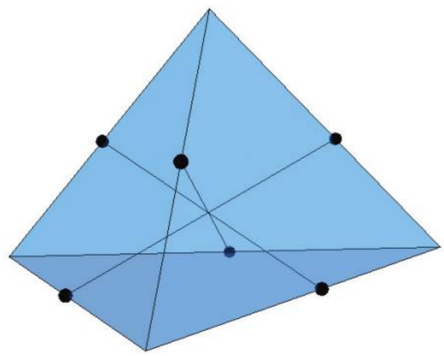

Figure 3 The bimedians of a tetrahedron

PROPOSITION 3. *For any tetrahedron, the sum of the squares of its three bimedians is equal to one fourth of the sum of the squares of its edges.*

In this note we provide a generalization of these three propositions for an arbitrary number of points.

The general formula

To understand our strategy, it will be useful to look at some simple proofs of Propositions 1, 2, and 3. We write the Euclidean distance between any two points A and B as $d(A, B)$. It can be expressed through the norm and the scalar product, as follows:

$$d(A, B) = \|A - B\| = \sqrt{(A - B) \cdot (A - B)}.$$

Using properties of the scalar product (and quite a bit of algebra) we can verify that, given three points A, B, C,

$$d\left(A, \frac{B+C}{2}\right)^2 + d\left(B, \frac{A+C}{2}\right)^2 + d\left(C, \frac{A+B}{2}\right)^2 = \frac{3}{4}\left[d(A, B)^2 + d(A, C)^2 + d(B, C)^2\right],$$

thus proving Proposition 1. Similarly, given four points A, B, C, D, we directly verify that

$$d\left(A, \frac{B+C+D}{3}\right)^2 + d\left(B, \frac{A+C+D}{3}\right)^2 + d\left(C, \frac{A+B+D}{3}\right)^2 + d\left(D, \frac{A+B+C}{3}\right)^2 = \frac{4}{9}\left[d(A, B)^2 + d(A, C)^2 + d(A, D)^2 + d(B, C)^2 + d(B, D)^2 + d(C, D)^2\right],$$

proving Proposition 2, and

$$d\left(\frac{A+B}{2}, \frac{C+D}{2}\right)^2 + d\left(\frac{A+C}{2}, \frac{B+D}{2}\right)^2 + d\left(\frac{A+D}{2}, \frac{B+C}{2}\right)^2$$
$$= \frac{1}{4}\Big[d(A,B)^2 + d(A,C)^2 + d(A,D)^2 + d(B,C)^2 + d(B,D)^2 + d(C,D)^2\Big],$$

proving Proposition 3.

In the last formula, we could have considered six bimedians, each one starting from one edge and joining it to the opposite edge. We only wrote three of them because the other ones coincide with these, two by two. To avoid possible misunderstandings in the sequel, it will be preferable to deal with means, rather than sums. In general, given a finite set of real numbers $\mathcal{S} = \{x_1, \dots, x_N\}$, we will use the notation

$$\text{Mean } \mathcal{S} = \frac{x_1 + \cdots + x_N}{N}.$$

Our aim is to find a generalization of the three formulas above to any set of n points. The idea is, first, to fix integers j and k satisfying $j \geq 1$, $k \geq 1$, and $j + k \leq n$. Then, from among the n points, take two distinct subsets, one consisting of j points and one consisting of k points. Compute the barycenter of each subset, and compute the squared distance between these two barycenters. Average over all possible choices of the two subsets. Our claim is that there is a constant α with the following property:

The mean of the squares of the distances between the couples of barycenters thus obtained is equal to the constant α multiplied by the mean of the squares of all segments joining the n points.

And this turns out to be true, as the following theorem states.

THEOREM 1. *Let n, j and k be three integers such that $j \geq 1$, $k \geq 1$, $j + k \leq n$, and set*

$$\alpha_{j,k} = \frac{j+k}{2jk}.$$

Then, for any given n points $A_1, A_2, \dots, A_n$,

$$\text{Mean}\left\{\begin{array}{c} d\left(\dfrac{A_{i_1} + \cdots + A_{i_j}}{j}, \dfrac{A_{i_{j+1}} + \cdots + A_{i_{j+k}}}{k}\right)^2 : \\ 1 \leq i_1 < \cdots < i_j \leq n,\ 1 \leq i_{j+1} < \cdots < i_{j+k} \leq n \\ \{i_1, \dots, i_j\} \cap \{i_{j+1}, \dots, i_{j+k}\} = \phi \end{array}\right\}$$
$$= \alpha_{j,k} \text{ Mean}\{d(A_p, A_q)^2 : 1 \leq p < q \leq n\}.$$

Concerning the above formula, we notice that the arithmetic mean on the left-hand side involves $\binom{n}{j} \cdot \binom{n-j}{k}$ terms, while on the right-hand side we have the *mean square distance* of the points $A_1, \dots, A_n$, which involves $\binom{n}{2}$ numbers. Remarkably, the constant α does not depend on n, but only on j and k.

Before carrying out the proof of the theorem, let us consider some particular situations.

The case $n = 3$, $j = 1$, $k = 2$ yields Proposition 1. On the other hand, taking $n = 4$, if $j = 1$ and $k = 3$ we get Proposition 2, while if $j = 2$ and $k = 2$ we obtain Proposition 3. In these cases, we have

$$\alpha_{1,2} = \frac{3}{4}, \quad \alpha_{1,3} = \frac{2}{3}, \quad \alpha_{2,2} = \frac{1}{2}.$$

More generally, let us define a *median*, in the general n points case, as the segment joining one of the points to the barycenter of the remaining $n - 1$ points. We then easily deduce the following generalization of Propositions 1 and 2.

COROLLARY 1. *Given n points, the sum of the squares of the n medians is equal to $n/(n-1)^2$ times the sum of the squares of all segments joining the n points.*

Let us now define a *bimedian*, in the general n points case, as the segment joining the midpoint of one segment to the barycenter of the remaining $n - 2$ points. Then, the following generalization of Proposition 3 holds.

COROLLARY 2. *Given n points, the sum of the squares of the $n(n-1)/2$ bimedians is equal to $n/(4n-8)$ times the sum of the squares of all segments joining the n points.*

Notice that, in the case $n = 4$, the six bimedians considered in the above corollary coincide two by two. This explains why, in this case, we now have *one half* of the sum of the squares of its edges, instead of *one fourth*, as stated in Proposition 3.

Let us mention that the results stated here hold in any real or complex inner product space.

Proof of the formula

We now go for the proof of Theorem 1. For simplicity, the proof will be carried out in the real case, but only minor modifications are needed if we consider a complex inner product space. We intend to prove that

$$\sum_{\{i_1,\ldots,i_{j+k}\}} \left\| \frac{A_{i_1} + \cdots + A_{i_j}}{j} - \frac{A_{i_{j+1}} + \cdots + A_{i_{j+k}}}{k} \right\|^2 = \nu_{n,j,k} \sum_{1 \le p < q \le n} \|A_p - A_q\|^2,$$

where

$$\nu_{n,j,k} = \frac{\binom{n}{j}\binom{n-j}{k}}{\binom{n}{2}}\, \alpha_{j,k},$$

the first sum being taken on all sequences of distinct indices $\{i_1, \ldots, i_{j+k}\}$ in $\{1, \ldots, n\}$ such that $1 \le i_1 < \cdots < i_j \le n$ and $1 \le i_{j+1} < \cdots < i_{j+k} \le n$.

It is easy to see that

$$\sum_{1 \le p < q \le n} \|A_p - A_q\|^2 = (n-1) \sum_{p=1}^{n} \|A_p\|^2 - 2 \sum_{1 \le p < q \le n} A_p \cdot A_q.$$

Let us now concentrate on the left-hand side of the identity. The sum appearing there, for symmetry reasons, will be developed as

$$\sum_{\{i_1,\ldots,i_{j+k}\}} \left\| \frac{A_{i_1} + \cdots + A_{i_j}}{j} - \frac{A_{i_{j+1}} + \cdots + A_{i_{j+k}}}{k} \right\|^2$$

$$= \beta_{n,j,k} \sum_{p=1}^{n} \|A_p\|^2 + \gamma_{n,j,k} \sum_{1 \le p < q \le n} A_p \cdot A_q,$$

where $\beta_{n,j,k}$ and $\gamma_{n,j,k}$ are constants, to be determined. In order to find the first one, let us compute, for instance, the coefficient of $\|A_1\|^2$. If A_1 belongs to the first group, then $\|A_1\|^2$ will have a factor $1/j^2$, and this may happen $\binom{n-1}{j-1} \cdot \binom{n-j}{k}$ times. On the other hand, if A_1 belongs to the second group, then $\|A_1\|^2$ will have a factor $1/k^2$, and this may happen $\binom{n-1}{k-1} \cdot \binom{n-k}{j}$ times. Then, summing the two and simplifying,

$$\begin{aligned}\beta_{n,j,k} &= \binom{n-1}{j-1}\binom{n-j}{k}\frac{1}{j^2} + \binom{n-1}{k-1}\binom{n-k}{j}\frac{1}{k^2} \\ &= \frac{(n-1)!}{j!\,k!\,(n-j-k)!}\,\frac{j+k}{jk} \\ &= (n-1)\nu_{n,j,k}.\end{aligned}$$

In order to find the value of $\gamma_{n,j,k}$, let us now compute, for instance, the coefficient of $A_1 \cdot A_2$. We distinguish four cases.

I. Assume $j \geq 2$ and $k \geq 2$. If A_1 and A_2 both belong to the first group, then $A_1 \cdot A_2$ will have a factor $2/j^2$, and this may happen $\binom{n-2}{j-2} \cdot \binom{n-j}{k}$ times. If A_1 belongs to the first group and A_2 to the second one, then $A_1 \cdot A_2$ will have a factor $-2/(jk)$, and this may happen $\binom{n-2}{j-1} \cdot \binom{n-j-1}{k-1}$ times. The same if A_1 belongs to the second group and A_2 to the first one. If A_1 and A_2 both belong to the second group, then $A_1 \cdot A_2$ will have a factor $2/k^2$, and this may happen $\binom{n-2}{k-2} \cdot \binom{n-k}{j}$ times. Summing up and simplifying,

$$\begin{aligned}\gamma_{n,j,k} &= \binom{n-2}{j-2}\binom{n-j}{k}\frac{2}{j^2} + 2\binom{n-2}{j-1}\binom{n-j-1}{k-1}\frac{-2}{jk} + \binom{n-2}{k-2}\binom{n-k}{j}\frac{2}{k^2} \\ &= -2\,\frac{(n-2)!}{j!\,k!\,(n-j-k)!}\,\frac{j+k}{jk} \\ &= -2\,\nu_{n,j,k}.\end{aligned}$$

II. Assume $j = 1$ and $k \geq 2$. If A_1 belongs to the first group and A_2 to the second one, then $A_1 \cdot A_2$ will have a factor $-2/k$, and this may happen $\binom{n-2}{k-1}$ times. The same if A_1 belongs to the second group and A_2 to the first one. If A_1 and A_2 both belong to the second group, then $A_1 \cdot A_2$ will have a factor $2/k^2$, and this may happen $\binom{n-2}{k-2} \cdot (n-k)$ times. Summing up,

$$\begin{aligned}\gamma_{n,1,k} &= 2\binom{n-2}{k-1}\frac{-2}{k} + \binom{n-2}{k-2}(n-k)\frac{2}{k^2} \\ &= -2\,\frac{(n-2)!}{k!\,(n-1-k)!}\,\frac{1+k}{k} \\ &= -2\,\nu_{n,1,k}.\end{aligned}$$

III. Assume $k = 1$ and $j \geq 2$. As in case II, we find

$$\gamma_{n,j,1} = -2\,\nu_{n,j,1}.$$

IV. Finally, assume $j = k = 1$. If A_1 belongs to the first group and A_2 to the second one, then $A_1 \cdot A_2$ will have a factor -2, and this may happen only once. The same if A_1 belongs to the second group and A_2 to the first one. Then,

$$\gamma_{n,1,1} = 2(-2) = -4 = -2\,\nu_{n,1,1}.$$

So, in all four cases, we have that $\gamma_{n,j,k} = -2\,\nu_{n,j,k}$. In conclusion, we see that

$$\begin{aligned}\sum_{\{i_1,\ldots,i_{j+k}\}} \left\| \frac{A_{i_1}+\cdots+A_{i_j}}{j} - \frac{A_{i_{j+1}}+\cdots+A_{i_{j+k}}}{k} \right\|^2 \\ &= (n-1)\nu_{n,j,k}\sum_{p=1}^{n}\|A_p\|^2 - 2\,\nu_{n,j,k}\sum_{1\le p<q\le n} A_p\cdot A_q \\ &= \nu_{n,j,k}\left[(n-1)\sum_{p=1}^{n}\|A_p\|^2 - 2\sum_{1\le p<q\le n} A_p\cdot A_q\right] \\ &= \nu_{n,j,k}\sum_{1\le p<q\le n}\|A_p - A_q\|^2,\end{aligned}$$

and the proof is completed.

A further example

We have seen how our formula applies to triangles and tetrahedra. We now illustrate another particular example, by considering a regular octahedron. The results in this section are developed as an illustration of our main formula. For comparison, they can also be obtained by more elementary methods.

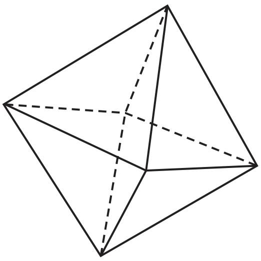

Figure 4 A regular octahedron

In this case we have six points $A_1, \ldots, A_6$, so $n = 6$. For simplicity, we consider only the cases when $j + k = 6$, and we write the formulas for $j = 1, 2, 3$. Let us denote by ℓ the length of the edges of the octahedron. First of all, we notice that the mean square distance of the vertices is

$$\frac{1}{15}\sum_{1\le p<q\le 6}\|A_p - A_q\|^2 = \frac{12\,\ell^2 + 3(\ell\sqrt{2})^2}{15} = \frac{6}{5}\,\ell^2.$$

To fix the ideas, assume that A_1, A_2, A_3 determine a face of the octahedron (i.e., an equilateral triangle), and let A_4, A_5, and A_6 be opposite to A_1, A_2, and A_3, respectively.

The case $j = 1$, $k = 5$. Let us denote by m the length of the six medians. Since $\alpha_{1,5} = \frac{3}{5}$, our formula gives

$$\frac{6\,m^2}{6} = \frac{3}{5}\cdot\frac{6}{5}\,\ell^2,$$

from which we derive

$$m = \frac{3}{5}\sqrt{2}\,\ell.$$

The case $j = 2, k = 4$. We are dealing with bimedians, i.e., the segments joining the midpoint of one edge to the barycenter of the remaining four points. Among these fifteen bimedians, twelve of them have a positive length, which we denote by b, while the remaining three have a zero length, being reduced to the center of the octahedron. Since $\alpha_{2,4} = \frac{3}{8}$, our formula says that

$$\frac{12\,b^2 + 3 \cdot 0^2}{15} = \frac{3}{8} \cdot \frac{6}{5}\,\ell^2,$$

whence

$$b = \frac{3}{4}\,\ell.$$

The case $j = 3, k = 3$. Here we take two groups of three vertices, and we want to compute the distances between their barycenters. However, we have to deal with two possible situations.

As we said above, the points A_1, A_2, A_3 determine one face, and the remaining three points A_4, A_5, A_6 determine the opposite face. There are eight such situations (where pairs of faces are involved), which coincide two by two. Let us denote by d the distance between these two faces.

On the other hand, the points A_1, A_2 and A_4, for instance, do not determine a face, but a right triangle, as well as the complementary points A_3, A_5, A_6. There are twelve such pairs of triangles, even if they coincide two by two. Let us denote by δ the distance between the barycenters of these two triangles. Using the symmetries of the regular octahedron, it is easy to see that

$$\delta = \frac{\ell\sqrt{2}}{3}.$$

Since $\alpha_{3,3} = \frac{1}{3}$, our formula tells us that

$$\frac{8\,d^2 + 12\,\delta^2}{20} = \frac{1}{3} \cdot \frac{6}{5}\,\ell^2,$$

and we deduce that

$$d = \sqrt{\frac{2}{3}}\,\ell.$$

Acknowledgment Thanks to my son Marcello for the drawings of the tetrahedra.

REFERENCES

1. N. A. Court, *Modern Pure Solid Geometry,* Macmillan, New York, 1935.
2. G. Y. Sosnow, A geometry problem, *Amer. Math. Monthly* **25** (1918) 122. http://dx.doi.org/10.2307/2971976

Summary We propose a general formula involving n points in an Euclidean space, which generalizes, on one hand, a well-known formula for the medians of a triangle and, on the other hand, two other formulas involving either the medians or the bimedians of a tetrahedron.

Generic Ellipses as Envelopes

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A curve C is said to be an *envelope* of a family of curves if each curve of the family is tangent to C. The astroid, described by the equation

$$\left(\frac{x}{k}\right)^{2/3} + \left(\frac{y}{k}\right)^{2/3} = 1, \quad k > 0,$$

is well known [**3**] as being an envelope of two different families of curves. The first is a family of line segments, as in FIGURE 1. A line segment of length k, sliding without slipping so that its endpoints lie on the coordinate axes (imagine a ladder sliding down a wall), remains tangent to an astroid. The astroid is also the envelope of a family of concentric ellipses (as in FIGURE 2), where the sum of the lengths of the axes is the constant $2k$.

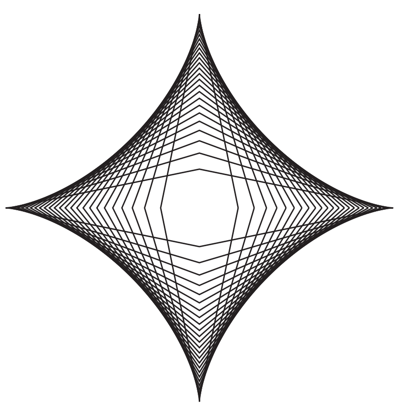

Figure 1

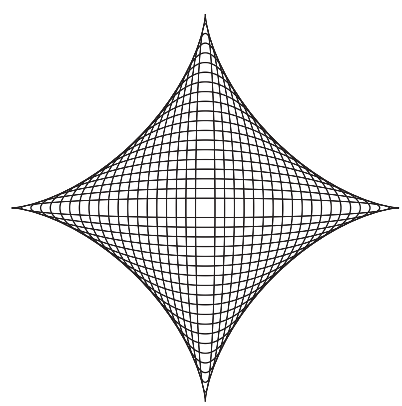

Figure 2

As it happens, there is a surprising connection between these two figures, which will be revealed in the main result of this paper. A related diagram is one produced by a typical grade school art student on a piece of graph paper: In the first quadrant, draw lines such that the sum of the x- and y-intercepts is some constant $k > 0$, then reflect to the other quadrants. This results in FIGURE 3, which can be described by

$$\left|\frac{x}{k}\right|^{1/2} + \left|\frac{y}{k}\right|^{1/2} = 1.$$

Elimination of roots from this equation when $x, y \geq 0$ results in

$$x^2 - 2xy + y^2 - 2kx - 2ky + k^2 = 0,$$

a parabola. Only the part of this parabola with both $x \leq k$ and $y \leq k$ is seen in the first quadrant of FIGURE 3.

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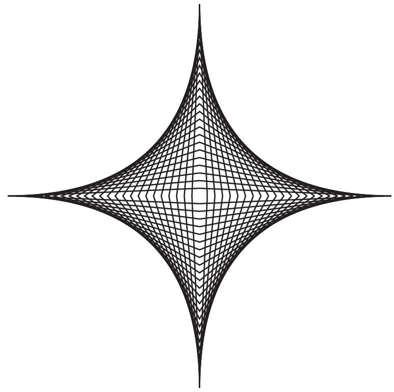

Figure 3

What do these figures have in common? For $a, b, m > 0$, define a *generic ellipse* $E(a, b; m)$ to be the set of points satisfying

$$\left|\frac{x}{a}\right|^m + \left|\frac{y}{b}\right|^m = 1.$$

We refer to the four points $(\pm a, 0)$ and $(0, \pm b)$ as the *vertices* of the generic ellipse. Note that the ellipse is smooth only when $m > 1$. When $m = 1$, the vertices are the four corners of a diamond, and they are cusps when $0 < m < 1$. We refer to E as an *m-ellipse*.

One more definition is needed to facilitate a discussion of the above figures. For $p \neq 0$, the *p-length of* (x, y) is defined to be

$$\|(x, y)\|_p = (|x|^p + |y|^p)^{1/p}. \tag{1}$$

(We avoid the term "norm" as we will have occasion to use this definition when $p < 1$.) In FIGURE 1, we see that the line segments, in sets of four, form 1-ellipses. To say that the lengths of these segments is constant is simply to say that $\|(a, b)\|_2$ is constant for members of this family of ellipses $E(a, b; 1)$.

In FIGURE 2, to say that the sum of the lengths of the axes of the concentric ellipses is constant is to say that that $\|(a, b)\|_1$ is constant for members of this family of ellipses $E(a, b; 2)$.

Certainly the symmetry of the parameters "1" and "2" in producing the same envelope—the astroid—is suggestive. But before a generalization can be proved, it is necessary to answer the following question: When are two generic ellipses $E_1(a_1, b_1; m_1)$ and $E_2(a_2, b_2; m_2)$ tangent?

Tangency of generic ellipses

First note that tangency at cusps can only occur if $a_1 = a_2$ or $b_1 = b_2$. When the ellipses are smooth, tangency at vertices can only occur if either $a_1 = a_2$ or $b_1 = b_2$. When $m_1 = m_2 > 1$, E_1 and E_2 can *only* be tangent at vertices (unless they are the same m_1-ellipse), as we will see later.

Turning our attention to tangency at points other than vertices, we now assume that $m_1 \neq m_2$. Due to symmetry, we consider the case when E_1 and E_2 are tangent at (x_0, y_0) in the first quadrant, so that $x_0, y_0 > 0$.

A routine calculation reveals that the tangent lines to E_1 and E_2 at (x_0, y_0) are given by

$$\frac{xx_0^{m_i-1}}{a_i^{m_i}} + \frac{yy_0^{m_i-1}}{b_i^{m_i}} = 1,$$

for $i = 1, 2$. Now E_1 is tangent to E_2 at (x_0, y_0) if these lines are the same, so that

$$\frac{x_0^{m_1-1}}{a_1^{m_1}} = \frac{x_0^{m_2-1}}{a_2^{m_2}}, \qquad \frac{y_0^{m_1-1}}{b_1^{m_1}} = \frac{y_0^{m_2-1}}{b_2^{m_2}}. \tag{2}$$

(Note that if $m_1 = m_2$, and if E_1 and E_2 were tangent at a point that is not a vertex, so that $x_0 \neq 0$ and $y_0 \neq 0$, then (2) would imply $a_1 = a_2$ and $b_1 = b_2$, and hence E_1 and E_2 would be identical.)

Now we may solve for x_0 and y_0 to determine the purported point of tangency:

$$(x_0, y_0) = \left(\left(\frac{a_1^{m_1}}{a_2^{m_2}}\right)^{\frac{1}{m_1-m_2}}, \left(\frac{b_1^{m_1}}{b_2^{m_2}}\right)^{\frac{1}{m_1-m_2}} \right). \tag{3}$$

Of course it is necessary that (x_0, y_0) lie on both E_1 and E_2; substituting into either of their equations yields

$$\left(\frac{a_1}{a_2}\right)^{\frac{m_1 m_2}{m_1-m_2}} + \left(\frac{b_1}{b_2}\right)^{\frac{m_1 m_2}{m_1-m_2}} = 1. \tag{4}$$

Thus (4) is a necessary and sufficient condition for E_1 and E_2 to be tangent at a point other than a vertex; and when they are tangent, the point of tangency in the first quadrant is given by (3). Moreover, as suggested by FIGURES 1–3, E_1 lies inside [outside] E_2 if $m_1 > m_2$ [$m_1 < m_2$] (except possibly for the vertices). This is not difficult to show using Hölder's inequality; details are included in the Appendix.

We are now ready to generalize and prove the observations about FIGURES 1 and 2 made earlier.

Main result

THEOREM. *Let an m_1-ellipse $E(a, b; m_1)$ be given, as well as $k > 0$ and $q \neq 0$. Suppose that $\|(a, b)\|_q = k$. Then this m_1-ellipse is tangent to the m_2-ellipse*

$$\left|\frac{x}{k}\right|^{m_2} + \left|\frac{y}{k}\right|^{m_2} = 1,$$

where m_2 is determined by

$$\frac{1}{m_2} = \frac{1}{m_1} + \frac{1}{q}. \tag{5}$$

This suggests that we may describe an m_2-ellipse as the envelope of a family of m_1-ellipses, as illustrated above. We will revisit FIGURES 1–3 in this context after giving the straightforward proof.

To begin the proof, let E, k, and q be as described. Since $\|(a, b)\|_q = k$, we have

$$a^q + b^q = k^q,$$

or equivalently,

$$\left(\frac{a}{k}\right)^q + \left(\frac{b}{k}\right)^q = 1. \tag{6}$$

Then by (4), this m_1-ellipse is tangent to $E(k, k; m_2)$ precisely when

$$\left(\frac{a}{k}\right)^{\frac{m_1 m_2}{m_1 - m_2}} + \left(\frac{b}{k}\right)^{\frac{m_1 m_2}{m_1 - m_2}} = 1. \tag{7}$$

Now, considering the family of generic ellipses

$$|x|^m + |y|^m = 1$$

for $m > 0$, it is evident that the point $(a/k, b/k)$ lies on precisely one such generic ellipse unless the point is a vertex, which is impossible when $a, b > 0$. Hence, comparison of (6) and (7) results in

$$q = \frac{m_1 m_2}{m_1 - m_2},$$

which is equivalent to (5). This completes the proof of the theorem.

We now revisit FIGURES 1–3. FIGURE 1 is an application of our theorem with $m_1 = 1$ and $q = 2$. In this case, we are creating a family of diamonds $E(a, b; 1)$ with the property that $\|(a, b)\|_2 = k$; that is, the length of the sides of the diamonds is k. This gives the family of generic ellipses, $E(a, \sqrt{k^2 - a^2}; 1)$. Since

$$m_2 = \frac{m_1 q}{m_1 + q} = \frac{2}{3}$$

in this case, all such diamonds are tangent to

$$\left|\frac{x}{k}\right|^{2/3} + \left|\frac{y}{k}\right|^{2/3} = 1,$$

so that the envelope of a family of such diamonds is an astroid.

We also see that FIGURE 2 is an application of our theorem with $m_1 = 2$ and $q = 1$. The symmetry of m_1 and q in (5) implies that an astroid is *also* an envelope of a family of ellipses. Thus, the envelope of the family of ellipses $E(a, k - a; 2)$ is again the astroid described above.

FIGURE 3 illustrates the case $m_1 = q = 1$, giving the family of generic ellipses $E(a, k - a; 1)$. This results in $m_2 = \frac{1}{2}$, so that the envelope of this family is

$$\left|\frac{x}{k}\right|^{1/2} + \left|\frac{y}{k}\right|^{1/2} = 1.$$

Note that interchanging m_1 and q yields nothing new in this case.

Further examples

We now illustrate the theorem with additional examples. In FIGURE 4, we have $m_1 = 2$ and $q = 8$, so that $m_2 = 8/5$ and the envelope

$$\left|\frac{x}{k}\right|^{8/5} + \left|\frac{y}{k}\right|^{8/5} = 1$$

is produced by a family of ellipses. Since the statement of the theorem is symmetric in m_1 and q, we may produce the same envelope using $m_1 = 8$ and $q = 2$; this is illustrated in FIGURE 5.

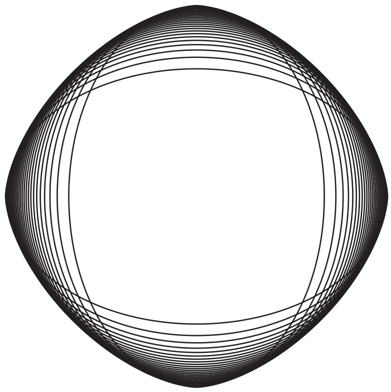

Figure 4

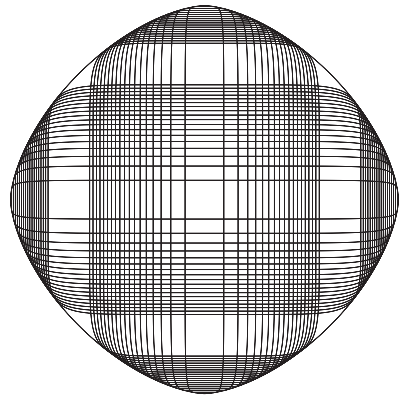

Figure 5

FIGURE 6 is produced with $m_1 = q = 2$, and so $m_2 = 1$. This family of ellipses is enveloped by

$$\left|\frac{x}{k}\right| + \left|\frac{y}{k}\right| = 1.$$

Note that this figure has no "mate," since interchanging m_1 and q produces nothing new.

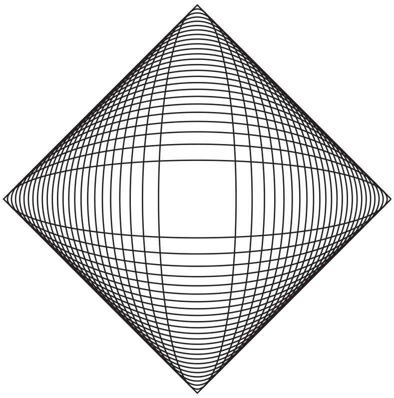

Figure 6

As remarked earlier, it need not be the case that $q \geq 1$. FIGURE 7 illustrates the case where $m_1 = 1$ and $q = 3/4$, so that the envelope

$$\left|\frac{x}{k}\right|^{3/7} + \left|\frac{y}{k}\right|^{3/7} = 1$$

is obtained. FIGURE 8 depicts its mate, with $m_1 = 3/4$ and $q = 1$.

Treating (1) as a purely algebraic statement, we may even allow $q < 0$. Some interesting envelopes result, as shown below. However, as we still require $m_1, m_2 > 0$,

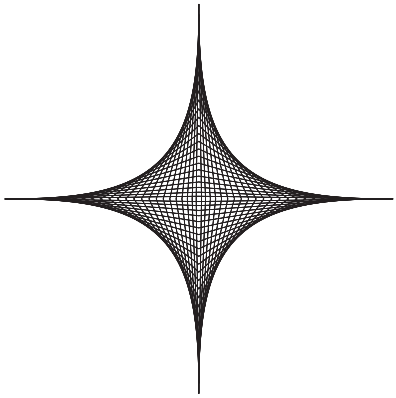

Figure 7

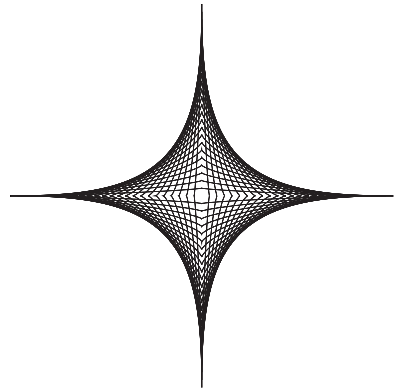

Figure 8

it follows from (5) that $m_1 < m_2$ in this case. Thus each m_1-ellipse lies *outside* the m_2-ellipse. This again follows from Hölder's inequality (see the Appendix). The careful reader will want to verify that the proof of the theorem remains valid when $q < 0$; while the family of curves $|x|^m + |y|^m = 1$ for $m < 0$ no longer consists of generic ellipses, it is still the case that no two distinct such curves intersect.

FIGURE 9 illustrates the case when $m_1 = 2$ and $q = -6$, so that $m_2 = 3$, while FIGURE 10 depicts the case $m_1 = 1$, $q = -6$, and $m_2 = 6/5$. Note that because $q < 0$ in these cases, m_1 and q cannot be interchanged to create a new family of ellipses.

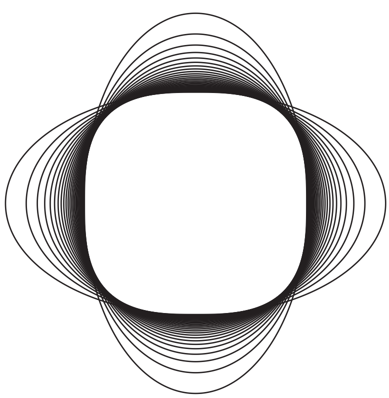

Figure 9

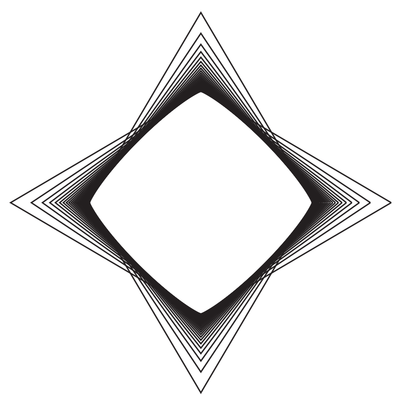

Figure 10

It was the beauty of envelopes that first attracted the author's attention. Perhaps gone are the days when envelopes were created with pen and straightedge. But the advent of computer graphics allows for the creation of some truly striking images. It is hoped that the few included here might inspire others to explore this sublime geometrical world.

Finally, we remark that all results easily extend to higher dimensions. The proofs were written in such a way that raising the dimension is simply a matter of adding coordinates in a list or terms in a summand. The essential nature of the calculations, however, remains the same.

Appendix: Inscription of generic ellipses

Hölder's inequality [**1**, p. 379] states that if $r > 0$ and $s > 0$ satisfy $1/r + 1/s = 1$, then for $n > 0$, and positive real numbers $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$, we have

$$\sum_{k=1}^{n} u_k v_k \le \left(\sum_{k=1}^{n} u_k^r\right)^{1/r} \left(\sum_{k=1}^{n} v_k^s\right)^{1/s}.$$

Moreover, equality holds precisely when

$$\frac{u_i^r}{v_i^s} = \frac{u_j^r}{v_j^s}, \qquad 1 \le i, j \le n.$$

We may generalize slightly in the case that $R, S, T > 0$, and $1/R + 1/S = 1/T$. Then for positive $U_1, \ldots, U_n$ and $V_1, \ldots, V_n$, we have

$$\left(\sum_{k=1}^{n} (U_k V_k)^T\right)^{1/T} \le \left(\sum_{k=1}^{n} U_k^R\right)^{1/R} \left(\sum_{k=1}^{n} V_k^S\right)^{1/S}.$$

Moreover, equality holds precisely when

$$\frac{U_i^R}{V_i^S} = \frac{U_j^R}{V_j^S}, \qquad 1 \le i, j \le n.$$

This follows directly from Hölder's inequality with $u_k = U_k^T$, $v_k = V_k^T$, $r = R/T$, and $s = S/T$. Note that it is not necessary that R, S, and T be at least 1.

In our case, $n = 2$. Assume that $x, y > 0$ and $q = m_1 m_2/(m_1 - m_2) > 0$ (so that $m_1 > m_2$), where the notations are as in the discussion preceding the statement of the theorem. Put

$$R = m_1, \quad S = q, \quad T = m_2, \quad U_1 = \frac{x}{a_1}, \quad U_2 = \frac{y}{b_1}, \quad V_1 = \frac{a_1}{a_2}, \quad V_2 = \frac{b_1}{b_2}.$$

We immediately get

$$\left(\left(\frac{x}{a_2}\right)^{m_2} + \left(\frac{y}{b_2}\right)^{m_2}\right)^{1/m_2} \le \left(\left(\frac{x}{a_1}\right)^{m_1} + \left(\frac{y}{b_1}\right)^{m_1}\right)^{1/m_1} \left(\left(\frac{a_1}{a_2}\right)^{q} + \left(\frac{b_1}{b_2}\right)^{q}\right)^{1/q}.$$

But the right-hand side of this inequality is 1, since (x, y) is on E_1 and (4) is valid. Thus, (x, y) is either on or inside E_2. It may readily be verified that equality holds precisely (x, y) is given by (3).

The case when $q < 0$ is handled similarly. Simply rewrite $1/m_2 = 1/m_1 + 1/q$ as $1/m_1 = 1/m_2 + 1/(-q)$ and apply the generalized inequality.

Acknowledgments The author would like to thank James Tao for his helpful comments.

REFERENCES

1. M. Berger, *Geometry I,* Springer, New York, 1987.
2. V. G. Boltyanskii, *Envelopes,* Macmillan, New York, 1964.
3. E. H. Lockwood, *A Book of Curves,* Cambridge University Press, Cambridge, 1963.

Summary The astroid is well known as an envelope both of a family of line segments and a family of ellipses. The relationship between these two families is investigated by asking when two generic ellipses are tangent to each other, where a generic ellipse is described by $|x/a|^m + |y/b|^m = 1$ for some $a, b, m > 0$. Results are illustrated with several examples of envelopes of families of generic ellipses.

Hermitian Conjugation

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The theoretical physicist Paul Dirac (1902–1984) represented quantum systems' states using angle brackets. He designated a state by $|\psi\rangle$, which he called a"ket." He called the ket's Hermitian conjugate $\langle\psi|$ a "bra." The inner product $\langle\psi|\psi\rangle$ involves a bra and a ket but no "c" between them.

His is one of many competing notations for the Hermitian conjugate.

How do you, in text, disclose
complex versions of transpose:
Signify them via star?
Broadcast them by means of bar?

Friends, Hermitian conjugation
has degenerate notation.

Algebraists, one might ask:
Why put asterisks to this task?
Use of symbols astronomic
yields a discord inharmonic:
In some contexts, "Multiply
minus-one times every *i*"
is the message meant for us
in a sign siderius.
If you favor this convention,
calculate with apprehension:
Proofs can crumple from omissions
of the matrix transposition.

Nearer Earth, then, we must seek
symbols for Monsieur Hermite.

Physicists prefer the dagger;
hence their students bleed and stagger,
dabbing eyes with tissues teary,
from exams on quantum theory.

If you study waves and packets—
if you write out angle brackets—
if you use Dirac's notation—
caveat: His formulation
of the "ket's" Hermitian double
is the "bra"—for prudes, some trouble.

Since we share no one convention,
symbol space has great dimensions.
Though I love variety,
we've overshot satiety.
Without dimension unity,
we suffer from disunity.

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Functions with Dense Graphs

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Most of us love an extreme example. In this note, we would like to adorn the "bouquet of discontinuous functions" [**1**] with our favorite flower: a function with a dense graph. Classical examples can be found in [**3**] and in chapter 9 of [**5**]. Here, we describe a new family of examples. Corresponding to each surjective function $r : \mathbb{N} \to \mathbb{Q}^+$ we construct a function $f_r : \mathbb{Q}^+ \to \mathbb{Q}^+$ with a graph dense in the first quadrant.

We let $\mathbb{N}$ denote the set of natural numbers $\{1, 2, 3, \ldots\}$, and $\mathbb{Q}^+$ denote the set of positive rational numbers. In topological conversations we say that a set D is *dense* in a topological space X when every nonempty open subset of X contains an element of D. When the topological space is $\mathbb{Q}^+$, this property can be phrased in terms of the open intervals

$$\mathcal{O}_{(a,b)} \equiv \{ x \mid a < x < b \},$$

the basic open subsets of $\mathbb{Q}^+$. A subset D of $\mathbb{Q}^+$ is dense if and only if $\mathcal{O}_{(a,b)} \cap D \neq \phi$ for all $a < b$ in $\mathbb{Q}^+$. When D is a subset of

$$\mathbb{Q}^+ \times \mathbb{Q}^+ \equiv \left\{ (x, y) \mid x, y \in \mathbb{Q}^+ \right\},$$

then D is dense in the first quadrant exactly when every nonempty open rectangle

$$\mathcal{O}_{(a,b)} \times \mathcal{O}_{(c,d)} \equiv \left\{ (x, y) \mid x \in \mathcal{O}_{(a,b)} \text{ and } y \in \mathcal{O}_{(c,d)} \right\}$$

contains elements of D. Of course, since the rational quadrant is dense in the real quadrant, such a set D is also a dense subset of the real first quadrant. When you have convinced yourself that a graph

$$\left\{ (x, f(x)) \mid x \in \mathbb{Q}^+ \right\}$$

is dense in the first quadrant if and only if, for each $a < b$ in $\mathbb{Q}^+$, $f(\mathcal{O}_{(a,b)})$ is dense in $\mathbb{Q}^+$, you begin to appreciate that functions with dense graphs are the extreme antithesis of continuous functions.

Assume that we are given a surjective function $r : \mathbb{N} \to \mathbb{Q}^+$, which we will call an *enumeration* of $\mathbb{Q}^+$. Define the function $f_r : \mathbb{Q}^+ \to \mathbb{Q}^+$ that maps the rational m/n (written in reduced form) to $r(m)$. We will prove that f_r has a graph that is dense in the first quadrant. With the classic enumeration

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \ldots,$$

the graph is approximated in FIGURE 1, where we have plotted the points $(x, f_r(x))$ for $x = i/10^4$ ($i = 1, \ldots, 10^4$). We encourage the reader to play with f_r for other enumerations r, like the fascinating ones in [**2**].

Math. Mag. **86** (2013) 366–369. doi:10.4169/math.mag.86.5.366. © Mathematical Association of America

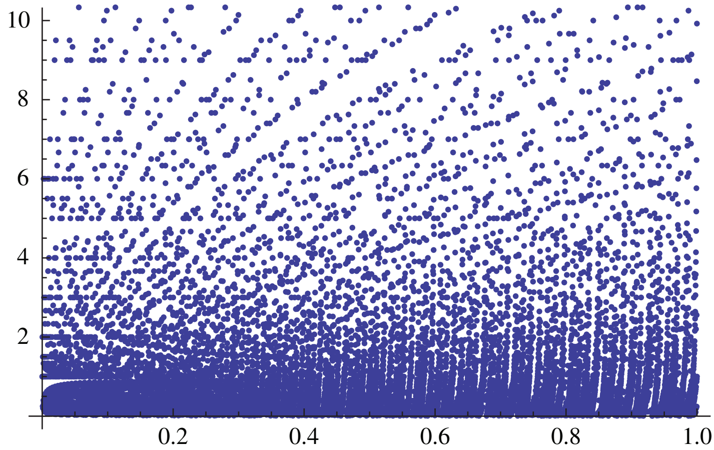

Figure 1 Some points in the graph of f_r with the classic enumeration

This project began when we found a completely elementary (though quite intricate) proof that the graph of f_r is dense in the case of the classic enumeration. But it turns out that the result is much more general, applying to all enumerations. For the general proof presented below, we rely on the prime number theorem, a calculus problem, and a short lemma.

The prime number theorem For each $x \in \mathbb{Q}^+$, let $\pi(x)$ denote the number of prime positive integers p less than or equal to x. The prime number theorem gives a handle on the rate of growth of π. A proof can be found in [**4**].

PRIME NUMBER THEOREM. *The function $\pi(x)$ behaves asymptotically like $x/\ln x$; that is,*

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln(x)} = 1.$$

A calculus problem By assigning the following problem, we give ourselves an excuse to state the prime number theorem to calculus students.

PROBLEM. *If $0 < a < b$, show that* $\lim_{x\to\infty} \dfrac{\pi(bx) - \pi(ax)}{x/\ln x} = b - a.$

The solution uses the prime number theorem. With with $c > 0$, we get

$$\begin{aligned} 1 = \lim_{x\to\infty} \frac{\pi(cx)}{cx/\ln(cx)} &= \frac{1}{c} \lim_{x\to\infty} \frac{\pi(cx)}{x/\ln x} \left(\frac{\ln(cx)}{\ln x} \right) \\ &= \frac{1}{c} \lim_{x\to\infty} \frac{\pi(cx)}{x/\ln x} \left(1 + \frac{\ln c}{\ln x} \right). \end{aligned}$$

The limit of the last factor is 1, so it follows that

$$\lim_{x\to\infty} \frac{\pi(cx)}{x/\ln(x)} = c,$$

which solves the calculus problem.

A short lemma The lemma tells us that we can find numbers where we need them that are relatively prime to m.

LEMMA. *Assume that $a, b \in \mathbb{Q}^+$ with $a < b$. There exists $N \in \mathbb{N}$ such that, for all integers $m > N$, there exist $p_m \in \mathcal{O}_{(am,bm)}$ relatively prime to m.*

Proof. Choose N_1 so that $aN_1 > 1$, then find $q > 1$ so that $(aN_1)^q > N_1$. The calculus problem implies that $\pi(bm) - \pi(am) \to \infty$ as $m \to \infty$, so there exists $N \geq N_1$ with

$$\pi(bm) - \pi(am) > q$$

for all $m > N$. This is the N of which we state the existence in the short lemma. We now prove that it does what was claimed of it.

Assume that $m > N$. Recalling $(aN_1)^q > N_1$ convinces us that $a^q N_1^{q-1} > 1$, and

$$a^q m^{q-1} > a^q N^{q-1} \geq a^q N_1^{q-1} > 1.$$

Multiply both sides of the inequality above by m to obtain $(am)^q > m$. Our choice of N ensures that the interval $\mathcal{O}_{(am,bm)}$ contains at least q primes $(p_i)_{i=1}^q$. Each of these primes satisfies $am < p_i$, so that $(am)^q < \Pi_{i=1}^q p_i$. If each p_i is a divisor of m $(i = 1, \dots, q)$, then we have $\Pi_{i=1}^q p_i \leq m$, contradicting the inequality

$$m < (am)^q < \Pi_{i=1}^q p_i.$$

Thus there exists a prime $p \in \{p_1, \dots, p_q\}$ such that p does not divide m. This prime number has the property that $p \in \mathcal{O}_{(am,bm)}$, and it is relatively prime to m, so we choose $p_m = p$ to establish our lemma. ■

THEOREM. *Assume that $r : \mathbb{N} \to \mathbb{Q}^+$ is an enumeration of the positive rational numbers, and define $f : \mathbb{Q}^+ \to \mathbb{Q}^+$, for reduced $m/n \in \mathbb{Q}^+$, by*

$$f\left(\frac{m}{n}\right) = r(m).$$

Then the graph of f is dense in the first quadrant.

Proof. To show the graph of f dense in the first quadrant, let an interval $\mathcal{O}_{(u,v)}$ in $\mathbb{Q}^+$ be given $(0 < u < v)$. We need to show that $f(\mathcal{O}_{(u,v)})$ is a dense subset of $\mathbb{Q}^+$. Apply the lemma to the numbers $a = 1/v$ and $b = 1/u$. We obtain N such that, for each $m > N$, we have p_m relatively prime to m with

$$\frac{m}{v} < p_m < \frac{m}{u}.$$

Consequently, we have

$$u < \frac{m}{p_m} < v$$

for all $m > N$. We conclude that $f(\mathcal{O}_{(u,v)})$ contains the set

$$T \equiv \left\{ f\left(\frac{m}{p_m}\right) \mid m > N \right\} = \{r(m) : m > N\}.$$

Since T contains all but finitely many points of $\mathbb{Q}^+$, it is dense in $\mathbb{Q}^+$. Therefore, $f(\mathcal{O}_{(u,v)})$ is dense in $\mathbb{Q}^+$, as required. ■

REFERENCES

1. G. Drago, P. D. Lamberti, and P. Toni, A "bouquet" of discontinuous functions for beginners in mathematical analysis, *Amer. Math. Monthly* **118** (2011) 799–810.
2. S. P. Glasby, Enumerating the rationals from left to right, *Amer. Math. Monthly* **118** (2011) 830–835.
3. G. Hamel, Eine Basis aller Zalen und die unstetigen Losungen der Funktionalgleichung: $f(x + y) = f(x) + f(y)$, *Math. Ann.* **60** (1905) 459–462. http://dx.doi.org/10.1007/BF01457624
4. M. France and G. Tenenbaum, *The Prime Numbers and Their Distribution*, Student Mathematical Library, Volume 6, American Mathematical Society, Providence, RI, 2000.
5. A. B. Kharazishvili, *Strange Functions in Real Analysis*, Pure and Applied Mathematics, Chapman & Hall, New York, 2006.

Summary We describe how to turn an enumeration of the rational numbers into a function with a dense graph.

ACROSS

1. Soft parts of fruits, as you might not prefer in juices
6. Oz. and kg.
9. Org. founded in 1915
12. Fencing swords
13. Mathematician William Feller or Branko Grunbaum, e.g.
15. ___ and outs
16. "I study networks and connections"
18. Alternative to a coffee break
19. Kissers
20. "Messiah" composer
21. CMU online education program
22. What Newton's method produces (abbr.)
23. Scientific American or American Scientist, e.g. (abbr.)
24. Opposite of NNW
25. Class of intracellular calcium channels found in animal muscle tissues
26. "I study shapes in space"
29. Encl. with a manuscript
30. Borden Dairy Company spokescow
31. Pants that go down to the shins
33. "I study how to count things"
36. Qualification, or cautionary detail
37. Beach
38. They come in India Pale and Amber varieties
39. "I study shapes and spaces"
43. Player that predates DVD/Blu-Ray players
44. Org. headquartered in Providence, RI
47. Button found on a remote for a 43-Across
48. ___-image or ___-conditioning
49. Where you'll find Fr. and Sp.
50. Aleph or Pi, for example
52. Classic Pontiac muscle cars
53. ___ carte
54. "I study certain measures"
56. You might get stuck in one, alas
57. One who prays to Vishnu
58. Al _____ (pasta specification)
59. When doubled, an African fly
60. Had lunch, say
61. Building block toys

DOWN

1. Partner with an eye patch and hook, perhaps
2. Revolt
3. Arrive at too quickly, as in a conclusion
4. Livens (up)
5. UNIX command for a secure remote login
6. Tiger on the links
7. Resets the scale to zero
8. Way to do your hair
9. Portion of a circle's circumference measuring less than 180 degrees
10. "I study the underpinnings of 33-Down"
11. Hopes (to)
13. Distress caused by embarrassment
14. Stimpy's cartoon pal
17. "Eureka!"
23. _____ toast
27. "Forever Alone" and "Scumbag Steve" and "Grumpy Cat" and all manner of other annoying internet popularities
28. Supply for bakers and brewers
29. Sales pitch
31. Oscar nominee from "Silver Lining's Playbook", as you'd see him listed in a phonebook
32. _____'s Impossibility Theorem: result concerning fair elections
33. "I study change"
34. Assign too nice a grade to, perhaps
35. American transcendentalist writer of "Walden" and "Civil Disobedience"
36. What you'll find on the walls in Lascaux, for instance
40. Choosing to join, with "in"
41. Cave, as stolen from Italian
42. Repetitively enthusiastic phrase of excitement
44. Wolfram _____
45. Type of scholarship for academic achievement
46. _____-Cech compactification
51. As yet unannounced, like a schedule
52. Delight, or a popular Fox TV show about a singing group
55. Vectorized and numerical programming code commonly used with large datasets, as in image processing

What Do You Study?

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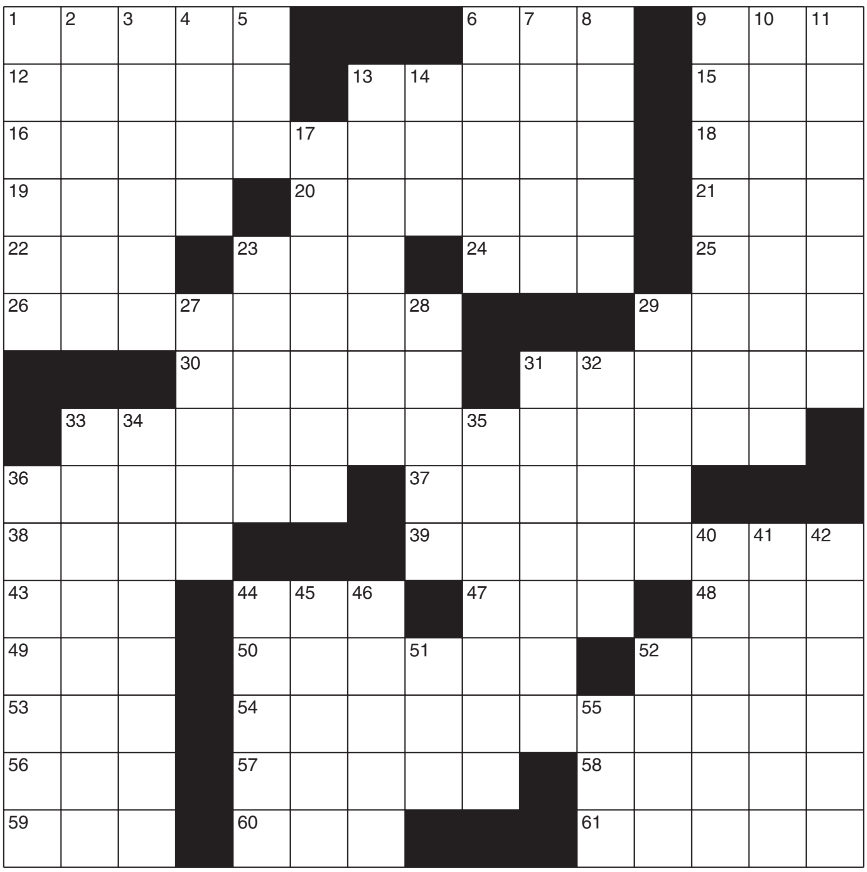

Clues are at left, on page 370. The solution is on page 397.

Extra copies of the puzzle, in both .pdf and .puz (AcrossLite) formats, can be found at the Magazine's website, or (temporarily) at http://www.mathematicsmagazine.org.

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The Period, Rank, and Order of the (a, b)-Fibonacci Sequence Mod m

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The Fibonacci sequence $F = 0, 1, 1, 2, 3, 5, 8, \dots$ has intrigued mathematicians for centuries, as it seems there is no end to its many surprising properties. Of particular interest to us are its properties when reduced under a modulus. It is well known, for example, that $F \pmod{m}$ is periodic, that the zeros are equally spaced, and that each period of $F \pmod{m}$ contains exactly 1, 2, or 4 zeros. We'll denote the period of $F \pmod{m}$ by $\pi(m)$. Formulas are known for computing $\pi(m)$ based on the prime factorization of m, but if p is prime, there is no formula for $\pi(p)$. However, certain divisibility relations hold: $\pi(p) \mid p - 1$ if $p \equiv \pm 1 \pmod{10}$, and $\pi(p) \mid 2(p + 1)$ if $p \equiv \pm 3 \pmod{10}$.

This paper arose from the realization that many of the modulo m properties of the Fibonacci sequence are also properties of a much larger class of sequences. Further, matrix methods offer elementary proofs for the general case that are no more difficult than for the Fibonacci sequence itself.

For integers a and b, we define the *(a, b)-Fibonacci sequence* F as the sequence with initial conditions $F_0 = 0$, $F_1 = 1$, that satisfies the general second-order linear recurrence relation $F_n = aF_{n-1} + bF_{n-2}$. So, for example, the $(1, 1)$-Fibonacci sequence is the classic case $F = 0, 1, 1, 2, 3, 5, \dots$, and the $(3, -2)$-Fibonacci sequence begins $0, 1, 3, 7, 15, 31, \dots$. In general, $F = 0, 1, a, a^2 + b, a^3 + 2ab, \dots$. In this article, we examine the behavior of the (a, b)-Fibonacci sequence under a modulus.

When reducing the (a, b)-Fibonacci sequence modulo m, we'll assume m is chosen so that $\gcd(b, m) = 1$. That way, the sequence is uniquely determined backward as well as forward. For instance, we can compute $F_{-1} \equiv b^{-1} \pmod{m}$. Modulo m, any pair of residues completely determines the sequence F, and there are finitely many pairs of residues, so F is periodic. We denote the period of $F \pmod{m}$ by $\pi(m)$.

The *rank of apparition,* or simply *rank,* of $F \pmod{m}$ is the least positive r such that $F_r \equiv 0 \pmod{m}$, and we denote the rank of $F \pmod{m}$ by $\alpha(m)$. If $F_{\alpha(m)+1} \equiv s \pmod{m}$, observe that the terms of F starting with index $\alpha(m)$, namely $0, s, as, (a^2 + b)s, \dots$, are exactly the initial terms of F multiplied by a factor of s.

Finally, we consider the *order* of $F \pmod{m}$, denoted by $\omega(m)$, and defined $\omega(m) = \pi(m)/\alpha(m)$. We shall see soon that $\omega(m)$ is always an integer, and that $\omega(m) = \mathrm{ord}_m(F_{\alpha(m)+1})$, the multiplicative order of $F_{\alpha(m)+1}$ modulo m. Other authors have not named this function, but its close connection with the multiplicative order of $F_{\alpha(m)+1}$ makes the name "order" seem reasonable.

Lucas studied the (a, b)-Fibonacci sequence extensively and in 1878 established foundational results on the rank [**9**, section XXV]. He assigned $\Delta = a^2 + 4b$ and deduced that if Δ is a quadratic residue (that is, a nonzero perfect square) mod p, then $\alpha(p) \mid p - 1$. Also, if Δ is a quadratic nonresidue (a residue that is not a perfect square), then $\alpha(p) \mid p + 1$. Finally, if $p \mid \Delta$, then $\alpha(p) = p$. These results were all

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obtained using the identity

$$2^{n-1}F_n = \binom{n}{1}a^{n-1} + \binom{n}{3}a^{n-3}\Delta + \binom{n}{5}a^{n-5}\Delta^2 + \cdots$$

Other authors followed by generalizing these results, or providing alternate proofs. See, e.g., [**1, 3, 6, 17**].

In 1960, Wall [**16**] produced results on the period of the (1, 1)-Fibonacci sequence and on the period of any integer sequence G satisfying $G_n = G_{n-1} + G_{n-2}$. Wall's paper seems to have renewed interest in the subject. In 1963, Vinson [**15**] and Robinson [**12**] both extended Wall's work; Vinson studied the order of the (1, 1)-Fibonacci sequence, and Robinson reproduced many results of Wall and Vinson, but with proofs greatly simplified by use of matrix methods. 1963 was also the year the *Fibonacci Quarterly* was established, and throughout the years many papers on the Fibonacci sequence modulo m have appeared there.

The study of generalized Fibonacci sequences under a modulus has continued in more recent years, and articles on the topic appear occasionally in this MAGAZINE. See, e.g., [**5, 8, 13, 14**]. See also [**7**] for a non-modular treatment of the (a, b)-Fibonacci sequence.

Through our study of the (a, b)-Fibonacci sequence modulo m, we hope to bring together many of the previous results, generalizing to the (a, b) case where necessary, and presenting them as a cohesive whole, using matrices as our main tool to supply elementary proofs.

Preliminaries

The matrix $A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ has the wonderful property that

$$A^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix},$$

where F here is the usual (1, 1)-Fibonacci sequence. This fact is extremely useful for the computation of very large Fibonacci numbers, and for finding and proving properties of F. Many authors use the matrix $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, but in this article we will follow the notation found in [**12**] and [**7**]; see [**4**] for more on the use of this and other matrices.

Let F denote the general (a, b)-Fibonacci sequence, let U denote the (a, b)-Fibonacci matrix below, and observe the form of U^n, which is easily confirmed by induction:

$$U = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}, \qquad U^n = \begin{bmatrix} bF_{n-1} & F_n \\ bF_n & F_{n+1} \end{bmatrix}.$$

Consequently, $U^{\pi(m)} \equiv I \pmod{m}$. Moreover, if $F_n \equiv 0$, then $F_{n-1} \equiv b^{-1}F_{n+1}$; thus, $U^{\alpha(m)} \equiv sI \pmod{m}$ for some integer s.

Observe that $\det U = -b$. Thus, $(-b)^{\pi(m)} = (\det U)^{\pi(m)} = \det U^{\pi(m)} \equiv 1 \pmod{m}$. So,

$$\operatorname{ord}_m(-b) \mid \pi(m).$$

This proves the well-known result for the (1, 1)-Fibonacci sequence that $\pi(m)$ is even for any $m > 2$.

The exponents n for which $U^n \equiv I \pmod{m}$ form a simple arithmetic progression ($U^0 \equiv I$, and if $U^i \equiv U^j \equiv I$, then $U^{i+j} \equiv I$). Thus,

$$U^n \equiv I \iff \pi(m) \mid n.$$

Similarly, the exponents n for which U^n is congruent to a scalar multiple of I form a simple arithmetic progression, and so

$$U^n \equiv sI \quad \text{for some} \quad s \in \mathbb{Z} \iff \alpha(m) \mid n.$$

From this we see that $\alpha(m) \mid \pi(m)$.

We defined the order of $F \pmod{m}$ as $\omega(m) = \pi(m)/\alpha(m)$, but $\omega(m)$ has another interpretation. If $U^{\alpha(m)} \equiv sI$, then $\mathrm{ord}_m(s)$ is the least positive value of k such that $U^{k\alpha(m)} \equiv I$. Consequently, $\mathrm{ord}_m(s)$ is the least positive k such that $\pi(m) \mid k\alpha(m)$. Clearly, the smallest such k is $\omega(m)$. Thus,

$$\text{If } U^{\alpha(m)} \equiv sI, \text{ then } \omega(m) = \mathrm{ord}_m(s).$$

Computing $\pi(m)$ and $\alpha(m)$

Much of our work in this paper is conducted with an eye toward constructing an algorithm that, given a, b, and m, will produce the period and rank of the (a, b)-Fibonacci sequence modulo m. The first step is recognizing that it is easy to compute $\pi(m)$ once we know $\pi(p^e)$ for all prime power factors p^e of m. The same idea holds for computing $\alpha(m)$.

The following theorem gives us the tool we need, and it is well known for the (1, 1)-Fibonacci sequence; see, e.g., [**15**]. In fact, our statement of the theorem for the (a, b) case is exactly the same as that for the (1, 1) case.

THEOREM 1. *Let brackets denote the least common multiple operation.*

(a) $\alpha([m_1, m_2]) = [\alpha(m_1), \alpha(m_2)]$

(b) $\pi([m_1, m_2]) = [\pi(m_1), \pi(m_2)]$

Proof. Let $m = [m_1, m_2]$.

Part (a). Let $\alpha = \alpha(m)$, $\alpha_1 = \alpha(m_1)$, and $\alpha_2 = \alpha(m_2)$. Since $F_\alpha \equiv 0 \pmod{m}$, we have $F_\alpha \equiv 0 \pmod{m_i}$ for each $i = 1, 2$. Thus, $\alpha_i \mid \alpha$ for each $i = 1, 2$ and we get $[\alpha_1, \alpha_2] \mid \alpha$.

Conversely, we know $F_{[\alpha_1,\alpha_2]} \equiv 0 \pmod{m_i}$ for each $i = 1, 2$, so $F_{[\alpha_1,\alpha_2]} \equiv 0 \pmod{m}$. Thus, $\alpha \mid [\alpha_1, \alpha_2]$.

Part (b). Let $\pi = \pi(m)$, $\pi_1 = \pi(m_1)$, and $\pi_2 = \pi(m_2)$. Since $U^\pi \equiv I \pmod{m}$, we have $U^\pi \equiv I \pmod{m_i}$ for each $i = 1, 2$. Thus, $\pi_i \mid \pi$ for each $i = 1, 2$ and we get $[\pi_1, \pi_2] \mid \pi$.

Conversely, we know $U^{[\pi_1,\pi_2]} - I \equiv 0 \pmod{m_i}$ for each $i = 1, 2$, and it follows that $U^{[\pi_1,\pi_2]} - I \equiv 0 \pmod{m}$. Thus, $\pi \mid [\pi_1, \pi_2]$. ∎

COROLLARY. *If $m_1 \mid m_2$, then $\alpha(m_1) \mid \alpha(m_2)$ and $\pi(m_1) \mid \pi(m_2)$.*

To apply the above theorem, suppose that $m = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$. Then $\alpha(m) = \left[\alpha(p_1^{e_1}), \alpha(p_2^{e_2}), \ldots, \alpha(p_k^{e_k})\right]$ and $\pi(m) = \left[\pi(p_1^{e_1}), \pi(p_2^{e_2}), \ldots, \pi(p_k^{e_k})\right]$.

Much more generally, Theorem 1 and its proof work for recurrence relations of any order, $S_n = a_1 S_{n-1} + a_2 S_{n-2} + \cdots + a_k S_{n-k}$. The theorem can even be used (with slight modification) when the modulus is not relatively prime to a_k (in which case the sequence cannot be uniquely determined for negative subscripts). See [**2**, p. 220] for a very general statement and interpretation of the theorem.

We now turn our attention to computing $\alpha(p^e)$ and $\pi(p^e)$, where p is a prime and e is a positive integer.

Computing $\pi(p^e)$ and $\alpha(p^e)$

It turns out that we can express $\pi(p^e)$ and $\alpha(p^e)$ in terms of $\pi(p)$ and $\alpha(p)$. The main result of this section, Theorem 2, shows exactly how to do that.

The proofs in this section follow those of [**12**], generalized to the (a, b) case. Again, the results here for general a and b are almost exactly those one finds in the literature for the $(1, 1)$-Fibonacci sequence; the slight differences are noted at the end of the section.

We begin by seeing how $\alpha(p^e)$ and $\alpha(p^{e+1})$ are related, and likewise for π.

PROPOSITION 1. *For any prime p and for any integer $e \geq 1$, $\alpha(p^{e+1}) = \alpha(p^e)$ or $p \cdot \alpha(p^e)$. Similarly, $\pi(p^{e+1}) = \pi(p^e)$ or $p \cdot \pi(p^e)$.*

Proof. Suppose that $U^n \equiv sI \pmod{p^e}$ for some integer s. Then $U^n = sI + p^e B$ for some matrix B. Then $U^{pn} = (sI + p^e B)^p = s^p I + \binom{p}{1} s^{p-1} p^e B + \cdots$, where every term after the first is divisible by p^{e+1}. Thus, $U^{pn} \equiv s^p I \pmod{p^{e+1}}$.

Now if $n = \alpha(p^e)$, then the conditions in the first line of this proof are satisfied, and we conclude that $\alpha(p^{e+1}) \mid p\alpha(p^e)$. But of course $\alpha(p^e) \mid \alpha(p^{e+1})$, and we conclude that $\alpha(p^{e+1}) = \alpha(p^e)$ or $p\alpha(p^e)$.

Similarly, if $n = \pi(p^e)$, then again the above conditions are satisfied (with $s = 1$) and we find that $\pi(p^{e+1}) = \pi(p^e)$ or $p\pi(p^e)$. ■

Thus, for each unit increase in e, $\alpha(p^e)$ either stays the same or increases by a factor of p. In fact, the next result shows that there is more going on: $\alpha(p^e)$ may stay constant initially, but once it starts to increase, it *must* continue increasing. The same is true of $\pi(p^e)$.

PROPOSITION 2. *Except for the single case $p = 2$ and $e = 1$, the following holds for any prime p and positive integer e.*

(a) *If $\alpha(p^e) \neq \alpha(p^{e+1})$, then $\alpha(p^{e+1}) \neq \alpha(p^{e+2})$.*
(b) *If $\pi(p^e) \neq \pi(p^{e+1})$, then $\pi(p^{e+1}) \neq \pi(p^{e+2})$.*

Proof. Suppose that $U^n \equiv sI \pmod{p^e}$ and that U^n is not congruent to any scalar multiple of $I \pmod{p^{e+1}}$. Then $U^n = sI + p^e B$, where $p^e B$ is not congruent to any scalar multiple of $I \pmod{p^{e+1}}$. Consequently, B is not congruent to any scalar multiple of $I \pmod{p}$.

Now $U^{pn} = (sI + p^e B)^p = s^p I + \binom{p}{1} s^{p-1} p^e B + \cdots$, and all terms replaced by the ellipsis are divisible by p^{e+2}. (This last statement is where we require any case other than $p = 2$ and $e = 1$.)

So, $U^{pn} \equiv s^p I \pmod{p^{e+1}}$. Moreover, since $p^{e+1} B$ is not congruent to any scalar multiple of $I \pmod{p^{e+2}}$, we also know that U^{pn} is not congruent to any scalar multiple of $I \pmod{p^{e+2}}$.

If $n = \alpha(p^e)$ and $\alpha(p^e) \neq \alpha(p^{e+1})$, then by Proposition 1, $pn = \alpha(p^{e+1})$ and the above argument implies $\alpha(p^{e+1}) \neq \alpha(p^{e+2})$.

Similarly, if $n = \pi(p^e)$ and $\pi(p^e) \neq \pi(p^{e+1})$, then $pn = \pi(p^{e+1})$ and the above argument implies $\pi(p^{e+1}) \neq \pi(p^{e+2})$. ■

The main result of this section is an immediate consequence and reformulation of the two preceding propositions. The last point of this theorem is deduced by inspection: modulo 2, we must have $b \equiv 1$, and so $F = 0, 1, 0, 1, \ldots$ (when $a \equiv 0$) or $F = 0, 1, 1, 0, 1, 1, \ldots$ (when $a \equiv 1$).

THEOREM 2. *Let an integer $e \geq 1$ be given.*

- *For odd p,*
  $\alpha(p^e) = p^{e-e'}\alpha(p)$, *where $1 \leq e' \leq e$ is maximal so that $\alpha(p^{e'}) = \alpha(p)$.*
  $\pi(p^e) = p^{e-e'}\pi(p)$, *where $1 \leq e' \leq e$ is maximal so that $\pi(p^{e'}) = \pi(p)$.*
- *For $p = 2$ and $e \geq 2$,*
  $\alpha(2^e) = 2^{e-e'}\alpha(4)$, *where $2 \leq e' \leq e$ is maximal so that $\alpha(2^{e'}) = \alpha(4)$.*
  $\pi(2^e) = 2^{e-e'}\pi(4)$, *where $2 \leq e' \leq e$ is maximal so that $\pi(2^{e'}) = \pi(4)$.*
- *Finally,*
  if a is odd, then $\alpha(2) = \pi(2) = 3$; if a is even, then $\alpha(2) = \pi(2) = 2$.

In the $(1, 1)$-Fibonacci sequence, it is an open problem whether any primes p exist such that $\pi(p^2) = \pi(p)$. Despite extensive searching, none have been found [**10**]. Even if such a p is found, there must exist *some* maximal e' such that $\pi(p^{e'}) = \pi(p)$, since no $(1, 1)$-Fibonacci number (other than F_0) is divisible by infinitely many powers of p.

However, in the more general (a, b) setting, we can find examples where $\pi(p^2) = \pi(p)$. Fix $a = 76$ and $b = 56$, and consider the behavior of $F \pmod{3^e}$ as e increases:

m	3	3^2	3^3	3^4	3^5	3^6	3^7	3^8
$\pi(m)$	6	6	18	54	162	486	1458	4374
$\alpha(m)$	3	3	3	3	3	3	9	27

We conclude that $\pi(3^e) = 3^{e-2} \cdot 6$ for $e \geq 2$, and $\alpha(3^e) = 3^{e-6} \cdot 3$ for $e \geq 6$.

We can also find examples where $\alpha(p^e)$ or $\pi(p^e)$ is constant for all e. If $a = 2$ and $b = -4$, then $F = 0, 1, 2, 0, -8, \ldots$. Consequently, for any odd prime p, $\alpha(p^e) = 3$ for all e and $\pi(p^e) = 3 \cdot \mathrm{ord}_{p^e}(-8)$. Finally, we might consider the case $a = 1$ and $b = -1$. In this situation, $F = 0, 1, 1, 0, -1, -1, 0, 1 \ldots$. Thus, for any odd prime p, $\alpha(p^e) = 3$ and $\pi(p^e) = 6$ for all positive integers e.

Computing $\pi(p)$ and $\alpha(p)$

Unfortunately, there are no explicit formulas for evaluating $\pi(p)$ and $\alpha(p)$. Perhaps this is not surprising, since $\pi(p)$ is the order of a matrix modulo p, and there is no explicit formula for computing the order of an integer modulo p. However, we do have divisibility relations.

By Theorem 2, we have $\alpha(2) = \pi(2) = 3$ or $\alpha(2) = \pi(2) = 2$. For the remainder of this section, we will assume that p is an odd prime.

The matrix U has characteristic polynomial $c(x) = x^2 - ax - b$. This polynomial has a root modulo p if the discriminant $a^2 + 4b$ is a perfect square modulo p. Specifically, if δ is an integer with the property that $\delta^2 \equiv a^2 + 4b \pmod{p}$, then the roots of $c(x) \pmod{p}$, as given by the quadratic formula, are $2^{-1}(a \pm \delta)$. The following theorem shows that we can gain insight into the divisibility properties of $\alpha(p)$ and $\pi(p)$ by considering the nature of $a^2 + 4b$.

THEOREM 3. *Let $\Delta = a^2 + 4b$ and let p be an odd prime such that $p \nmid b$. Then modulo p,*

(a) *if Δ is a (nonzero) quadratic residue, then $\alpha(p) \mid p - 1$ and $\pi(p) \mid p - 1$.*

(b) *if Δ is a quadratic nonresidue, then $\alpha(p) \mid p + 1$ and $\pi(p) \mid (p + 1)\mathrm{ord}_p(-b)$; also, except in the case $b \equiv -1$, $\pi(p) \nmid p + 1$.*

(c) *if $\Delta \equiv 0$, then $\alpha(p) = p$ and $\pi(p) = p \cdot \mathrm{ord}_p(2^{-1}a)$.*

Proof. For parts (a) and (b), we sketch the proofs found in [**5**].

(a) Suppose that Δ is a quadratic residue, modulo p. Then the characteristic polynomial $c(x)$ of U has two distinct roots, call them λ_1 and λ_2. Thus, U is diagonalizable and can be written $U \equiv PDP^{-1}$, where P is the matrix with eigenvectors as columns and

$$D \equiv \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Applying Fermat's Little Theorem, we find $D^{p-1} \equiv I$. Thus, $U^{p-1} \equiv I$ and part (a) of the theorem follows.

(b) Suppose that Δ is a quadratic nonresidue, modulo p. In this case, we switch our view from working with integers and congruences to working within the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. The polynomial $c(x)$ is irreducible in $\mathbb{F}_p$, but we can create a field extension $\mathbb{F}_p[\gamma]$ where γ has the property that $c(\gamma) = 0$. It is shown easily in [**5**] that $\gamma^p \neq \gamma$ and $c(\gamma^p) = 0$; thus $c(x)$ has two distinct roots in $\mathbb{F}_p[\gamma]$. So U is diagonalizable in $\mathbb{F}_p[\gamma]$ and can be written $U = PDP^{-1}$ for some matrix P and

$$D = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma^p \end{bmatrix}.$$

Since $(x - \gamma)(x - \gamma^p) = x^2 - ax - b$, we have $\gamma^{p+1} = -b$. Moreover, we observe that $(\gamma^p)^{p+1} = (-b)^p = -b$, with the final equality due to Fermat's Little Theorem. Thus, $U^{p+1} = PD^{p+1}P^{-1} = P\left[\begin{smallmatrix} -b & 0 \\ 0 & -b \end{smallmatrix}\right]P^{-1} = \left[\begin{smallmatrix} -b & 0 \\ 0 & -b \end{smallmatrix}\right]$. As a result, $\alpha(p) \mid p+1$, and unless $b \equiv -1 \pmod{p}$, $\pi(p) \nmid p+1$. It also follows that $U^{(p+1)\mathrm{ord}_p(-b)} = I$ and so $\pi(p) \mid (p+1)\mathrm{ord}_p(-b)$.

(c) Suppose that $\Delta \equiv 0 \pmod{p}$. Then $c(x)$ has a repeated root, $2^{-1}a$ (and $a \not\equiv 0$, otherwise $p \mid b$, a contradiction). In this case, U is not diagonalizable, but we can put U into Jordan form: $U = PJP^{-1}$ for some invertible P. Below, we see the form of J and of J^n.

$$J \equiv \begin{bmatrix} 2^{-1}a & 1 \\ 0 & 2^{-1}a \end{bmatrix}, \qquad J^n \equiv \begin{bmatrix} (2^{-1}a)^n & n(2^{-1}a)^{n-1} \\ 0 & (2^{-1}a)^n \end{bmatrix}.$$

Since $U^n \equiv PJ^nP^{-1}$, and since scalar multiples of I commute with any matrix, we find $U^n \equiv sI$ for some integer s if and only if $J^n \equiv sI$. So, considering J^n above, $\alpha(p)$ is the least integer n such that $n(2^{-1}a)^{n-1} \equiv 0 \pmod{p}$. Since $a \not\equiv 0 \pmod{p}$, we conclude that $\alpha(p) = p$.

Working modulo p, we obtain the following.

$$\begin{aligned} U^n \equiv I &\iff J^n \equiv I \\ &\iff (2^{-1}a)^n \equiv 1 \text{ and } n(2^{-1}a)^{n-1} \equiv 0 \\ &\iff \mathrm{ord}_p(2^{-1}a) \mid n \text{ and } p \mid n \\ &\iff \mathrm{lcm}[\mathrm{ord}_p(2^{-1}a), p] \mid n \\ &\iff p \cdot \mathrm{ord}_p(2^{-1}a) \mid n. \end{aligned}$$

By the above, $\pi(p) = p \cdot \mathrm{ord}_p(2^{-1}a)$. ∎

The proof of part (a) also shows that if λ_1 and λ_2 are the roots of $x^2 - ax - b$, then $\pi(m) = \mathrm{lcm}[\mathrm{ord}_p(\lambda_1), \mathrm{ord}_p(\lambda_2)]$.

We've not seen the part (c) result that $\pi(p) = p \cdot \mathrm{ord}_p(2^{-1}a)$ in the literature. However, the fact that $\alpha(p) = p$ is deduced by Lucas [**9**]. Our proof for (c) appears to be novel. It is curious that when $p \mid \Delta$, we have an explicit equality statement for $\pi(p)$ (albeit in terms of $\mathrm{ord}_p(2^{-1}a)$, which must be calculated).

For the standard $a = 1, b = 1$ situation, $\Delta = 5$. Using the law of quadratic reciprocity, we can find that 5 is a quadratic residue when $p \equiv \pm 1 \pmod{10}$ and 5 is a quadratic nonresidue when $p \equiv \pm 3 \pmod{10}$.

Finally, we note that combining Theorem 3 with the fact that $\mathrm{ord}_p(-b) \mid \pi(p)$ significantly narrows the possible values of $\pi(p)$, and can aid with a computer search for $\pi(p)$.

Properties of $\omega(m)$

The previous theorem showed how $\pi(p)$ and $\alpha(p)$ are related to the modulus, p. In this final section, we consider the relationship between $\alpha(m)$ and $\pi(m)$, as expressed by the function $\omega(m) = \pi(m)/\alpha(m)$. One of the most surprising things about the (1, 1)-Fibonacci sequence modulo m is that $\omega(m) = 1$, 2, or 4, no matter the value of m or the size of $\pi(m)$. Generally, however, for a fixed a and b, Theorem 4(a) shows us that $\omega(m)$ can take on infinitely many values as m varies. The following theorem and proof generalize those found in [**12**].

THEOREM 4.

(a) $\omega(m) \mid 2 \cdot \mathrm{ord}_m(-b)$

(b) $\pi(m) = (1 \textit{ or } 2) \cdot \mathrm{lcm}[\alpha(m), \mathrm{ord}_m(-b)]$

Proof. Suppose $U^{\alpha(m)} \equiv \left[\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right]$. Comparing determinants, we get $s^2 \equiv (-b)^{\alpha(m)}$. Raising both sides to $\mathrm{ord}_m(-b)$ yields $s^{2\cdot\mathrm{ord}_m(-b)} \equiv 1$. Since $\mathrm{ord}_s(m) = \omega(m)$, part (a) follows.

For part (b), we note that s^2 and $(-b)^{\alpha(m)}$, being congruent modulo m, have the same multiplicative order modulo m, namely,

$$\frac{\mathrm{ord}_m(s)}{\gcd(2, \mathrm{ord}_m(s))} = \frac{\mathrm{ord}_m(-b)}{\gcd(\alpha(m), \mathrm{ord}_m(-b))}.$$

Substituting $\omega(m)$ for $\mathrm{ord}_m(s)$ and cross-multiplying yields

$$\omega(m) \cdot \gcd(\alpha(m), \mathrm{ord}_m(-b)) = \gcd(2, \omega(m)) \cdot \mathrm{ord}_m(-b).$$

As a consequence,

$$\omega(m) = (1 \text{ or } 2)\frac{\mathrm{ord}_m(-b)}{\gcd(\alpha(m), \mathrm{ord}_m(-b))}.$$

Multiplying both sides of the equation by $\alpha(m)$ produces part (b) of the theorem.

$$\pi(m) = \omega(m)\alpha(m) = (1 \text{ or } 2) \cdot \mathrm{lcm}[\alpha(m), \mathrm{ord}_m(-b)]. \qquad \blacksquare$$

Theorem 4(a) gives us the interesting result that if $b = -1$, then $\omega(m)$ is always 1 or 2. We saw in the Preliminaries section that $\alpha(m) \mid \pi(m)$ and $\mathrm{ord}_m(-b) \mid \pi(m)$, so it immediately follows that $\mathrm{lcm}[\alpha(m), \mathrm{ord}_m(-b)] \mid \pi(m)$; part (b) of the above theorem makes this divisibility relation much more precise. Finally, we note that part (b) provides a very quick computation of $\pi(m)$, if $\alpha(m)$ and $\mathrm{ord}_m(-b)$ are known.

In the $a = 1, b = 1$ case, $\omega(p^e) = \omega(p)$ for any odd prime p [**11**, p. 38]. That is, π and α move in "lock step" with each other: $\pi(p^e) = \pi(p^{e+1}) \iff \alpha(p^e) = \alpha(p^{e+1})$. Our final theorem shows something similar for the general a, b setting. In the general setting, however, we find $\pi(p^e) = \pi(p^{e+1}) \Longrightarrow \alpha(p^e) = \alpha(p^{e+1})$ but the converse does not hold. Almost always, as e grows, $\omega(p^e)$ eventually becomes constant.

THEOREM 5.

(a) *Let p be an odd prime. If there exists an e such that $\alpha(p^e) = \alpha(p)$, but $\alpha(p^{e+1}) \neq \alpha(p)$, then $\omega(p^{e+i}) = \omega(p^e)$ for all $i \geq 0$.*

(b) *If there exists an e such that $\alpha(2^e) = \alpha(4)$, but $\alpha(2^{e+1}) \neq \alpha(4)$, then $\omega(2^{e+1+i}) = \omega(2^{e+1})$ for all $i \geq 0$.*

Proof. (a) Given $\alpha(p^e) = \alpha(p)$ and $\alpha(p^{e+1}) \neq \alpha(p)$, we apply Proposition 1 and Proposition 2(a) to conclude that $\alpha(p^{e+1}) = p\alpha(p)$. Assume for contradiction that $\pi(p^{e+1}) = \pi(p^e)$. Then, applying Proposition 2(b), we find that $\pi(2^{e+1}) = \pi(p)$. Thus,

$$\omega(p^{e+1}) = \frac{\pi(p^{e+1})}{\alpha(p^{e+1})} = \frac{\pi(p)}{p\alpha(p)} = \frac{\omega(p)}{p}.$$

Therefore, $p \mid \omega(p)$. But by Theorem 4(a), $\omega(p) \mid 2 \cdot \mathrm{ord}_p(-b)$. Since p is odd, $p \mid \mathrm{ord}_p(-b)$. But this is clearly a contradiction since $\mathrm{ord}_p(-b) \mid p - 1$.

Thus, our assumption was wrong, $\pi(p^{e+1}) \neq \pi(p^e)$, and so $\omega(p^e) = \omega(p^{e+1}) = \omega(p^{e+2}) = \cdots$.

The proof for part (b) is similar. Given $\alpha(2^e) = \alpha(4)$ and $\alpha(2^{e+1}) \neq \alpha(4)$, we conclude that $\alpha(2^{e+2}) = 4\alpha(4)$. Assume for contradiction that $\pi(2^{e+2}) = \pi(2^{e+1})$; then $\pi(2^{e+2}) = \pi(4)$. Thus,

$$\omega(2^{e+2}) = \frac{\pi(2^{e+2})}{\alpha(2^{e+2})} = \frac{\pi(4)}{4\alpha(4)} = \frac{\omega(4)}{4} = \frac{1 \text{ or } 2}{4}.$$

The last equality above is due to inspection: Since $\omega(2) = 1$, $\omega(4) = 1$ or 2. But $\omega(2^{e+2})$ must be an integer, so a contradiction has been found.

Thus, $\pi(2^{e+2}) \neq \pi(2^{e+1})$, and so $\omega(2^{e+1}) = \omega(2^{e+2}) = \omega(2^{e+3}) = \cdots$. ∎

The hypothesis in Theorem 5, that $\alpha(p^e) \neq \alpha(p)$ for some e, is almost always satisfied. In fact, if $\alpha(p^e) = \alpha(p)$ for all e, then we must have $F_{\alpha(p)} = 0$ (*equality*, not just congruence), a very strong requirement indeed. We previously noted the case $a = 2, b = -4$ in which $\alpha(p^e) = 3$ for all positive e, but $\pi(p^e)$ grows as e increases. In this case, $\omega(p^e)$ increases without bound.

For the $(1, 1)$-Fibonacci sequence, we noted that p odd implies $\omega(p^e)$ is constant as e grows. In the general a and b situation, more interesting behavior can be observed. Consider again the example $a = 76$ and $b = 56$, and observe the behavior of F (mod 3^e) as e increases:

m	3	3^2	3^3	3^4	3^5	3^6	3^7	3^8
$\pi(m)$	6	6	18	54	162	486	1458	4374
$\alpha(m)$	3	3	3	3	3	3	9	27
$\omega(m)$	2	2	6	18	54	162	162	162

In the above table, we see that $\omega(3^e)$ is initially constant, and then grows for a few terms before eventually stabilizing at 162.

We admit that the behavior of $\omega(p^e)$, when generalized from the $(1, 1)$ case to the general (a, b) case, loses some of its simple elegance. On the other hand, we are reminded once again of the many fascinating properties these sequences hold, and our imagination is stirred to try to understand them even better.

Acknowledgment I would like to extend my sincere appreciation to Josh Ide who, as an undergraduate student several years ago, spent many hours in my office discussing the Fibonacci sequence and modular arithmetic. As

a teacher, I hope to motivate students and ignite their curiosity about mathematics. However, it was Josh, through his insight and dedication, who inspired me to dig deeper into this topic.

REFERENCES

1. R. D. Carmichael, On sequences of integers defined by recurrence relations, *Quart. J. Math.* **41** (1920) 343–372.
2. Paul Cull, Mary Flahive, and Robby Robson, *Difference Equations,* Springer, New York, 2005.
3. H. T. Engstrom, On sequences defined by linear recurrence relations, *Trans. Amer. Math. Soc.* **33** (1931) 210–218. http://dx.doi.org/10.1090/S0002-9947-1931-1501585-5
4. H. W. Gould, A history of the Fibonacci Q-matrix and a higher-dimensional problem, *Fib. Quart.* **19**(3) (1981) 250–257.
5. Sanjai Gupta, Parousia Rockstroh, and Francis Edward Su, Splitting fields and periods of Fibonacci sequences modulo primes, *Math. Mag.* **85** (2012) 130–135. http://dx.doi.org/10.4169/math.mag.85.2.130
6. Marshall Hall, Divisors of second-order sequences, *Bull. Amer. Math. Soc.* **43** (1937) 78–80. http://dx.doi.org/10.1090/S0002-9904-1937-06497-4
7. Dan Kalman and Robert Mena, The Fibonacci numbers—exposed, *Math. Mag.* **76** (2003) 167–181. http://dx.doi.org/10.2307/3219318
8. Hua-Chieh Li, On second-order linear recurrence sequences: Wall and Wyler revisited, *Fib. Quart.* **37** (1999) 342–349.
9. Lucas, Eduoard, Théorie des fonctions numériques simplement périodiques, *Amer. J. of Math.* **1** (1878) 184–240, 289–321. http://dx.doi.org/10.2307/2369308
10. R. J. McIntosh and E. L. Roettger, A search for Fibonacci-Wieferich and Wolstenholme primes, *Math. Comp.* **76** (2007) 2087–2094. http://dx.doi.org/10.1090/S0025-5718-07-01955-2
11. Marc Renault, *Properties of the Fibonacci Sequence Under Various Moduli,* Master's thesis, Wake Forest University, May 1996. Available at http://webspace.ship.edu/msrenault/fibonacci/FibThesis.pdf.
12. D. W. Robinson, The Fibonacci matrix modulo m, *Fib. Quart.* **1** (1963) 29–36.
13. Dominic Vella and Alfred Vella, Cycles in the generalized Fibonacci sequence modulo a prime, *Math. Mag.* **74** (2002) 294–299.
14. Dominic Vella and Alfred Vella, Calculating exact cycle lengths in the generalized Fibonacci sequence modulo p, *Math. Gaz.* **90** (2006) 70–76.
15. John Vinson, The Relation of the Period Modulo m to the Rank of Apparition of m in the Fibonacci Sequence, *Fib. Quart.* **1** (1963) 37–45.
16. D. D. Wall, Fibonacci series modulo m, *Amer. Math. Monthly* **67** (1960) 525–532. http://dx.doi.org/10.2307/2309169
17. Morgan Ward, Prime divisors of second-order recurring sequences, *Duke Math. J.* **21** (1954) 607–614. http://dx.doi.org/10.1215/S0012-7094-54-02163-8

Summary For given integers a and b, we consider the (a, b)-Fibonacci sequence F defined by $F_0 = 0$, $F_1 = 1$, and $F_n = aF_{n-1} + bF_{n-2}$. Given $m \geq 2$ relatively prime to b, $F \pmod m$ is periodic with period denoted $\pi(m)$. The rank of $F \pmod m$, denoted $\alpha(m)$, is the least positive r such that $F_r \equiv 0 \pmod m$, and the order of $F \pmod m$, denoted $\omega(m)$, is $\pi(m)/\alpha(m)$. In this article, we pull together results on $\pi(m)$, $\alpha(m)$, and $\omega(m)$ from the classic case $a = 1$, $b = 1$, and generalize their proofs to accommodate arbitrary integers a and b. Matrix methods are used extensively to provide elementary proofs.

PROBLEMS

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PROPOSALS

To be considered for publication, solutions should be received by May 1, 2014.

1931. *Proposed by John E. Wetzel, University of Illinois at Urbana-Champaign, Urbana, IL.*

Find with proof the radius of the smallest disk that can cover every simple closed curve of unit length.

1932. *Proposed by Michel Bataille, Rouen, France.*

In 3-space, let R_O, R_ℓ, and R_P be the reflections by a point O, a line ℓ, and a plane P, respectively. Characterize the configurations for which $R_O \circ R_\ell \circ R_P$ is a rotation; if such is the case, show that $R_\ell \circ R_O \circ R_P$ and $R_O \circ R_P \circ R_\ell$ are rotations whose axes are parallel to the axis of $R_O \circ R_\ell \circ R_P$.

1933. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Let $n \geq 2$ be a natural number. Calculate

$$\int_0^1 \int_0^1 \cdots \int_0^1 \ln(1 - x_1 x_2 \cdots x_n) dx_1\, dx_2\, \cdots dx_n.$$

Math. Mag. **86** (2013) 381–387. doi:10.4169/math.mag.86.5.381.

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Solutions should be written in a style appropriate for this MAGAZINE.

Solutions and new proposals should be mailed to Bernardo M. Ábrego, Problems Editor, Department of Mathematics, California State University, Northridge, 18111 Nordhoff St, Northridge, CA 91330-8313, or mailed electronically (ideally as a LaTeX or pdf file) to mathmagproblems@csun.edu. All communications, written or electronic, should include **on each page** the reader's name, full address, and an e-mail address and/or FAX number.

1934. *Proposed by Greg Oman, University of Colorado, Colorado Springs, CO.*

Let R be an infinite ring (not assumed commutative or to contain an identity). Consider the following two conditions:

(a) R has zero divisors; that is, there exist nonzero elements $s, t \in R$ such that $st = 0$.

(b) There exist nonzero elements $s, t \in R$ such that $st = 0$. Further, $Rs \neq \{0\}$ and $Rt \neq \{0\}$ (here, $Rs := \{rs : r \in R\}$).

Does (a) imply that R possesses a nonzero left ideal I such that R and R/I have the same cardinality? Does the answer change if we assume (b) instead?

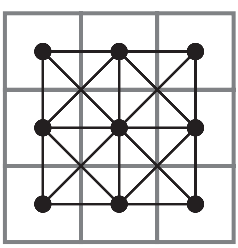
1935. *Proposed by Stan Wagon, Macalester College, St. Paul, MN.*

Let $K_{m,n}$ be the graph on the vertex set $\{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$, where the vertex (m_1, n_1) is connected to the vertex (m_2, n_2) if the king piece in chess can move from the square (m_1, n_1) to the square (m_2, n_2). Define $Q_{m,n}$ accordingly for the possible queen moves on an $m \times n$ chessboard.

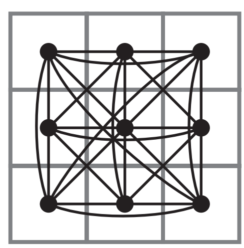
(a) For which pairs (m, n) is $K_{m,n}$ perfect?

(b) For which pairs (m, n) is $Q_{m,n}$ perfect?

Note: A graph is *perfect* if neither the graph nor its complement has a chordless odd cycle of length 5 or more.



$K_{3,3}$



$Q_{3,3}$

Quickies

Answers to the Quickies are on page 387.

Q1035. *Proposed by Jacek Fabrykowski and Thomas Smotzer, Youngstown State University, Youngstown, OH.*

Prove that there exists a triangle with sides of lengths a, b, and c if and only if a, b, and c are positive real numbers such that $2a^2b^2 + 2b^2c^2 + 2c^2a^2 > a^4 + b^4 + c^4$.

Q1036. *Proposed by Michael W. Botsko, Saint Vincent College, Latrobe, PA.*

Let $C = \{f : f \text{ is a continuous real-valued function defined on } \mathbb{R}\}$. In addition, let $D = \{f : f \text{ is a differentiable real-valued function on } \mathbb{R}\}$. Note that both C and D are rings under the usual definitions of addition and multiplication. Finally, let S be any subring of D containing a nonconstant linear function. Prove that C and S are not isomorphic.

Solutions

Proportional areas **December 2012**

1906. *Proposed by Yagub N. Aliyev, Department of Mathematics, Qafqaz University, Khyrdalan, Azerbaijan.*

Let ABC be an equilateral triangle and X a point in its interior such that $\angle CXA = 2\pi/3$. Suppose that the lines AX and BX intersect $\overline{BC}$ and $\overline{AC}$ at D and E, respectively. Prove that

$$\frac{1}{[\text{Area}(ABD)]^2} + \frac{1}{[\text{Area}(ADC)]^2} = \frac{1}{[\text{Area}(BDE)]^2 + [\text{Area}(ADE)]^2}.$$

Solution by Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Australia)
Let CX intersect AB in F. Since $\angle CXA = 2\pi/3$, then $\angle XCA = \pi/3 - \angle CAX = \angle XAB$. Hence, $\triangle CAF$ and $\triangle ABD$ are congruent, and thus $AF = BD$. We may assume that $AB = BC = CA = 1$ and let $AF = BD = u$ and $AE = w$. By Ceva's Theorem, we have that

$$\frac{u}{1-u} \cdot \frac{u}{1-u} \cdot \frac{1-w}{w} = 1,$$

which implies that

$$w = \frac{u^2}{(1-u)^2 + u^2}. \tag{1}$$

Denote by $[XYZ]$ the area of $\triangle XYZ$. We have that

$$\begin{aligned}
[ABD] &= u[ABC], \\
[ADC] &= (1-u)[ABC], \\
[BDE] &= (1-w)[ABD] = (1-w)u[ABC], \text{ and} \\
[ADE] &= (1-u)\,[ABE] = (1-u)w[ABC].
\end{aligned}$$

So, it is sufficient to show that

$$\frac{1}{u^2} + \frac{1}{(1-u)^2} = \frac{1}{u^2(1-w)^2 + (1-u)^2w^2}.$$

Using equation (1), we get

$$u^2(1-w)^2 + (1-u)^2w^2 = \frac{u^2(1-u)^2}{(1-u)^2 + u^2} = \frac{1}{\frac{1}{u^2} + \frac{1}{(1-u)^2}},$$

which completes the proof.

Also solved by George Apostolopoulos (Greece), Herb Bailey, Michel Bataille (France), Robert Calcaterra, Chip Curtis, Prithwijit De (India), Dmitry Fleischman, Mowaffaq Hajja (Jordan), Hyeon u Kim (Korea) and Dong Ho Shin (Korea), Omran Kouba (Syria), Elias Lampakis (Greece), Kee-Wai Lau (China), Donald Jay Moore, Peter Nüesch (Switzerland), Joel Schlosberg, Achilleas Sinefakopoulos (Greece), Irina Stallion, Haohao Wang and Jerzy Wojdylo, John Zacharias, and the proposer.

Rings with multiplicative finite order **December 2012**

1907. *Proposed by Daniel Lopez Aguayo, student, Institute of Mathematics, UNAM, Morelia, Mexico.*

Let R be a commutative ring with 1 such that there exists an integer n, $n \geq 2$, with the property that $r^n = r$ for all $r \in R$. Find with proof the intersection of all maximal ideals of R.

I. *Solution by Nicholas C. Singer, Annandale, VA.*
Denote the intersection by $J \supseteq \{0\}$. Suppose that $x \in J$. If $1 - x^{n-1}$ were not a unit, then $1 - x^{n-1} \in M$ for some maximal ideal M. Now $x \in M$ since $x \in J$ and J is the intersection of all the maximal ideals. Thus $1 = (1 - x^{n-1}) + x \cdot x^{n-2} \in M$, which is a contradiction. Then $1 - x^{n-1}$ is a unit, so there is a $z \in R$ with $(1 - x^{n-1})z = 1$. Hence $x = x \cdot 1 = x(1 - x^{n-1})z = (x - x^n)z = 0$. Thus $J = \{0\}$.

II. *Solution by Robert Calcaterra, University of Wisconsin–Platteville, Platteville, WI.*
Let J be the intersection of all the maximal ideals of R; that is, J is the Jacobson radical of R. Let $c \in J$. Since $c^n = c$, it follows that $c^{2(n-1)} = c^{2n-2} = c^n \cdot c^{n-2} = c \cdot c^{n-2} = c^{n-1}$. But no nonzero element of the Jacobson radical can be an idempotent, and thus $c^{n-1} = 0$. Hence, $c = c^n = c \cdot c^{n-1} = 0$. It follows that $J = \{0\}$ and the proof is complete.

Editor's Note. Many solvers used the fact that if x belongs to the Jacobson radical, then $1 - xy$ is invertible for all $y \in R$. This can be easily proved as follows: If $1 - xy$ is not invertible, then $1 - xy \in M$ for some maximal ideal M. Because $x \in J \subseteq M$, it follows that $1 = (1 - xy) + xy \in M$, which is a contradiction.

Now the fact stated in Calcaterra's solution can be verified as follows: If $x \in J$ and $x^2 = x$, then there is $y \in R$ such that $y(1 - x) = 1$. It follows that $x = y(1 - x)x = y(x - x^2) = 0$.

Calcaterra and Sordo Viera note that the conclusion still holds if the integer n in the hypotheses varies from one element of the ring to another. Calcaterra further notes that the result also holds in noncommutative rings if we consider the intersection of maximal right (or left) ideals instead of maximal ideals.

Also solved by George Apostolopoulos (Greece), Michel Bataille (France), Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Australia), Paul Budney, Bruce S. Burdick, Ahmad Habil (Syria), Francisco Perdomo (Spain) and Ángel Plaza (Spain), Achilleas Sinefakopoulos (Greece), Luis Sordo Vieira, Traian Viteam (Chile), Albert Whitcomb, and the proposer.

Inequalities for triangle measurements **December 2012**

1908. *Proposed by D. M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania.*

Let ABC be an arbitrary triangle with $a = BC$, $b = AC$, and $c = AB$. Denote by s, r, and R, the semiperimeter, inradius, and circumradius of $\triangle ABC$, respectively. Prove that if m and n are positive real numbers, then the following inequalities hold:

(a) $$\frac{a}{mb + nc} + \frac{b}{mc + na} + \frac{c}{ma + nb} \geq \frac{4s^2}{(m + n)(s^2 + r^2 + 4Rr)},$$

(b) $$\frac{1}{(ma + nb)^2} + \frac{1}{(mb + nc)^2} + \frac{1}{(mc + na)^2} \geq \frac{27}{4(m + n)^2 s^2}.$$

Solution by Achilleas Sinefakopoulos, M. N. Raptou Private High-School, Larissa, Greece.

First recall that $a+b+c=2s$ and that $ab+bc+ca=s^2+r^2+4Rr$.

(a) By the Cauchy–Schwarz inequality, it follows that

$$\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}\geq\frac{(a+b+c)^2}{x+y+z}$$

for any $x, y, z>0$. Setting $x=a(mb+nc)$, $y=b(mc+na)$, and $z=c(ma+nb)$, and noting that $x+y+z=(m+n)(ab+bc+ca)$, we obtain

$$\begin{aligned}\frac{a^2}{a(mb+nc)}+\frac{b^2}{b(mc+na)}+\frac{c^2}{c(ma+nb)}&\geq\frac{(a+b+c)^2}{(m+n)(ab+bc+ca)}\\&=\frac{(2s)^2}{(m+n)(s^2+r^2+4Rr)}.\end{aligned}$$

Inequality (a) follows immediately.

(b) From the Arithmetic Mean–Geometric Mean inequality, it follows that

$$(x+y+z)^2\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)\geq(3\sqrt[3]{xyz})^2\cdot\frac{3}{\sqrt[3]{x^2y^2z^2}}=27$$

for any $x, y, z>0$. Setting $x=ma+nb$, $y=mb+nc$, and $z=mc+na$, and noting that $x+y+z=(m+n)(a+b+c)=2(m+n)s$, we obtain

$$(2(m+n)s)^2\left(\frac{1}{(ma+nb)^2}+\frac{1}{(mb+nc)^2}+\frac{1}{(mc+na)^2}\right)\geq 27.$$

Inequality (b) follows immediately. The proof is complete.

Editor's Note. Some solvers showed a derivation of the known equation $ab+bc+ca=s^2+r^2+4Rr$ along the following lines: The area of $\triangle ABC$ equals $abc/4R=rs=(s(s-a)(s-b)(s-c))^{1/2}$. Thus $abc/s=4Rr$ and

$$\begin{aligned}r^2&=(s-a)(s-b)(s-c)/s=s^2-s(a+b+c)+(ab+bc+ca)-abc/s\\&=s^2-2s^2+(ab+bc+ca)-4Rr=-s^2-4Rr+(ab+bc+ca).\end{aligned}$$

Also solved by Arkady Alt, George Apostolopoulos (Greece), Michel Bataille (France), Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Australia), Minh Can, Chip Curtis, Bishal Deb (India) and Manjil P. Saikia (India), Michael Goldenberg and Mark Kaplan, John G. Heuver (Canada), Omran Kouba (Syria), Elias Lampakis (Greece),Paolo Perfetti (Italy), Peter Nüesch (Switzerland), Michael Vowe (Switzerland), and the proposer.

Back-and-forth for numerable real dense sets **December 2012**

1909. *Proposed by Howard Cary Morris, STCC, Memphis, TN.*

Let A and B be two countably infinite subsets of the interval $(0, 1)$.

(a) Show that if A and B are dense in $(0, 1)$, then there is a bijective function $f : A\to B$ such that f is strictly increasing on A, i.e., for every $a_1, a_2\in A$, $f(a_1)<f(a_2)$ whenever $a_1<a_2$.

(b) Show that such a function does not necessarily exist if A and B are not dense in $(0, 1)$.

Solution by Bob Tomper, Mathematics Department, University of North Dakota.
For part (a) we will construct a map $g : A \cup \{0, 1\} \to B \cup \{0, 1\}$ and then f will be the restriction of g to A. Let $a_0 = b_0 = 0$, $a_1 = b_1 = 1$, and let $\{a_n\}_{n=2}^{\infty}$ and $\{b_n\}_{n=2}^{\infty}$ be enumerations of A and B, respectively. We define g recursively, as follows: First, let $g(a_0) = b_0$ and $g(a_1) = b_1$. Next, suppose that g has been defined for $a_0, a_1, \ldots, a_k$, with values $g(a_j) = b_{i_j}$ for $j = 0, \ldots, k$. This defines k disjoint open subintervals $I_1, I_2, \ldots, I_k$ subdividing $(0, 1)$, with each $I_j = (a_m, a_n)$ for some $m, n \in \{0, 1, \ldots, k\}$, and corresponding open subintervals $J_j = (g(a_m), g(a_n))$, which subdivide $(0, 1)$. To define $g(a_{k+1})$, observe that a_{k+1} must lie in I_{j_0}, for some $j_0 \in \{1, \ldots, k\}$. Note that $B \cap J_{j_0} \neq \emptyset$, since B is dense in $(0, 1)$. Define $g(a_{k+1})$ to be the member of B with smallest index that lies in J_{j_0}.

The function g is injective, since we do not repeat any B values in choosing $g(a_{k+1})$. To show that g is surjective, consider a member b_n of B. Either b_n is one of the values $g(a_0), \ldots, g(a_n)$, or b_n lies in one of the intervals $J_1, \ldots, J_n$, say J_j. Since A is dense in $(0, 1)$, there is a smallest m greater than n such that $a_m \in I_j$, and therefore we will have $g(a_m) = b_n$.

For part (b), let $A = \{1 - 1/n\}_{n=2}^{\infty}$ and $B = \{1/n\}_{n=2}^{\infty}$. If $f : A \to B$ is strictly increasing, then $f(1/2) < f(2/3) < f(3/4) < \cdots$, which would imply the existence of infinitely many members of the sequence B greater than $f(1/2)$, which is clearly impossible.

Also solved by Jeremy F. Alm, George Apostolopoulos (Greece), John Atkins, Robert Calcaterra, Alex Chichester and Patrick Daniels, Dmitry Fleischman, Jerrold W. Grossman, Mowaffaq Hajja (Jordan), Eugene A. Herman, Elias Lampakis (Greece), Sean McIlroy (Canada), Valerian Nita, Northwestern University Math Problem Solving Group, Savan K. Patel (India) and Sanjay K. Patel (India), José M. Pacheco (Spain) and Ángel Plaza (Spain), Kenneth A. Ross, Achilleas Sinefakopoulos (Greece), John Zacharias, and the proposer.

A sequence of definite integrals — December 2012

1910. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Let $f : [a, b] \to [0, \infty)$ be a continuous function. Show that the following limit exists and calculate its value:

$$\lim_{n\to\infty} n \left[\sqrt[n]{\int_a^b (f(x))^{n+1} dx} - \sqrt[n]{\int_a^b (f(x))^n dx} \right].$$

Solution by Michel Bataille, Rouen, France.
For every positive integer n, let $I_n = \int_a^b (f(x))^n dx$ and $U_n = n(\sqrt[n]{I_{n+1}} - \sqrt[n]{I_n})$. If M denotes the maximum value of the continuous function f on $[a, b]$, we show that $\lim_{n\to\infty} U_n = 0$ if $M = 0$ and $\lim_{n\to\infty} U_n = M \ln(M)$ if $M > 0$. If $M = 0$, then $f(x) = 0$ for all $x \in [a, b]$, hence $I_n = I_{n+1} = 0 = U_n$ for all n and so $\lim_{n\to\infty} U_n = 0$. From now on, we suppose that $M > 0$; note that it follows that $I_n > 0$ for all $n \in \mathbb{N}$ (since f is continuous).

Because $(f(x))^n \leq M^n$ for all $x \in [a, b]$, it follows that $\sqrt[n]{I_n} \leq (b - a)^{1/n} M$, hence $\limsup_{n\to\infty} \sqrt[n]{I_n} \leq M$ (because $\lim_{n\to\infty}(b - a)^{1/n} = 1$). Let ε be an arbitrary positive real number. Since $M = f(x_0)$ for some $x_0 \in [a, b]$ and f is continuous, we have $f(x) \geq M - \varepsilon$ for x in a subinterval $[\alpha, \beta]$ of $[a, b]$ and it follows that $\sqrt[n]{I_n} \geq (\beta - \alpha)^{1/n}(M - \varepsilon)$, whence $\liminf_{n\to\infty} \sqrt[n]{I_n} \geq M - \varepsilon$. Thus $\liminf_{n\to\infty} \sqrt[n]{I_n} \geq M$ and thus

$$\lim_{n\to\infty} \sqrt[n]{I_n} = M. \tag{1}$$

We observe that $(f(x))^{n+1} \le M(f(x))^n$ for all $x \in [a,b]$ and so $I_{n+1} \le M \cdot I_n$. Moreover, we have

$$\frac{I_{n+2}}{I_{n+1}} - \frac{I_{n+1}}{I_n} = \frac{I_n \cdot I_{n+2} - (I_{n+1})^2}{I_n \cdot I_{n+1}} \ge 0$$

because $I_n \cdot I_{n+2} - (I_{n+1})^2 \ge 0$ by the Cauchy–Schwarz inequality. It follows that the sequence $\{I_{n+1}/I_n\}$ is nondecreasing and bounded above, hence convergent. Let L denote its limit (note that $L \ge I_2/I_1 > 0$). Then the sequence $\{\ln(I_{n+1}) - \ln(I_n)\}$ is convergent with limit $\ln(L)$ and the same is true of its Cesaro mean, so that $\{\frac{1}{n}\ln(I_n)\} = \{\ln(\sqrt[n]{I_n})\}$ converges with limit $\ln(L)$ as well. Using (1), it follows that $\ln(M) = \ln(L)$ and $L = M$.

Finally,

$$U_n = n\sqrt[n]{I_n}\left(\left(\frac{I_{n+1}}{I_n}\right)^{1/n} - 1\right) = n\sqrt[n]{I_n}\left(\exp\left(\frac{1}{n}\ln\left(\frac{I_{n+1}}{I_n}\right)\right) - 1\right)$$

and since $\lim_{n\to\infty} \frac{1}{n}\ln(I_{n+1}/I_n) = 0$, then, as $n \to \infty$,

$$\exp\left(\frac{1}{n}\ln\left(\frac{I_{n+1}}{I_n}\right)\right) - 1 \sim \frac{1}{n}\ln\left(\frac{I_{n+1}}{I_n}\right),$$

and thus

$$U_n \sim n\sqrt[n]{I_n} \cdot \frac{1}{n}\ln\left(\frac{I_{n+1}}{I_n}\right) = \sqrt[n]{I_n} \cdot \ln\left(\frac{I_{n+1}}{I_n}\right).$$

The result follows from (1) and $\lim_{n\to\infty} I_{n+1}/I_n = L = M$.

Also solved by Arkady Alt, George Apostolopoulos (Greece), Bruce S. Burdick, Hongwei Chen, Omran Kouba (Syria), Elias Lampakis (Greece), Paolo Perfetti (Italy), Tomas Persson (Sweden) and Mikael P. Sundqvist (Sweden), J. G. Simmonds, Achilleas Sinefakopoulos (Greece), Nicholas C. Singer, Haohao Wang and Yanping Xia, and the proposer. There were two incorrect or incomplete solutions.

Answers

Solutions to the Quickies from page 382.

A1035. Observe that $D = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4$ factors as

$$D = (a+b+c)(-a+b+c)(a-b+c)(a+b-c).$$

Suppose that $0 < a \le b \le c$. Note that the first three factors on the right-hand side of the equation are positive. By the triangle inequality, a triangle with sides a, b, and c exists if and only if $a + b - c > 0$; that is, if and only if $D > 0$.

A1036. Suppose that there exists $\phi : C \to S$, an isomorphism. Let g be the nonconstant linear function in S and write $g(x) = ax + b$. Let f be that function in C such that $\phi(f) = g$. Let $h = \sqrt[3]{f}$ and note that h is continuous on $\mathbb{R}$ and $(\phi(h))^3 = \phi(h^3) = \phi(f) = g = ax + b$. Thus $\phi(h)$ is the function $\sqrt[3]{ax+b}$, which is not differentiable at $x = -b/a$. Therefore $\phi(h)$ is not in S, which is a contradiction. This proves that C and S are not isomorphic.

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles, books, and other materials are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Henle, Michael, *Which Numbers Are Real?*, MAA, 2012; x + 219 pp, $53 ($43 to MAA members). ISBN 978-0-88385-777-9.

There are more numbers than most students dream of or even learn about in calculus. This book lists axioms for the real numbers, then constructs the reals both from sequences (Cantor) and from cuts (Dedekind). A section on "multidimensional numbers" captures the complex numbers and quaternions (but not octonions); relaxing other axioms leads to constructions of the constructive reals, the hyperreals, and the surreals. Much of the theory is developed through exercises (no solutions given). This book could serve as a prequel/sequel to a course in advanced calculus, or as a stand-alone for a course in "foundations."

Brown, Nicholas J.L., Alan D. Sokal, and Harris L. Friedman. Wishful thinking: The critical positivity ratio, *American Psychologist* 68 (2013) (in press), `http://arxiv.org/pdf/1307.7006.pdf`.

Fredrickson, Barbara L., and Losada, M.F., Positive affect and the complex dynamics of human flourishing, *American Psychologist* 60 (7) (2005) 678–686.

Fredrickson, Barbara L., Updated thinking on positivity ratios. *American Psychologist* 68 (2013) (in press), `http://cds.web.unc.edu/files/2013/07/Fredrickson_2013.pdf`.

Rotondaro, Vinnie, Nick Brown smelled bull, `http://narrative.ly/pieces-of-mind/nick-brown-smelled-bull/`.

The book reviewed above was concerned with which numbers are real; authors Brown et al. here bring up a similar question in the arena of applied mathematics: *Which applications are real?* They critique a psychology paper by Fredrickson and Losada (as well as two previous papers by Losada) as a travesty of mathematical modeling. Fredrickson and Losada assert that "flourishing mental health" needs a "critical positivity ratio," of positive emotions to negative emotions, greater than 2.9013. The basis for their assertion is a claimed fitting of empirical data to Lorenz's model for atmospheric convection, using (curiously) exactly the same values for the parameters as Saltzman and Lorenz used to illustrate the possibility of chaos and the Lorenz attractor. Brown et al. provide a useful checklist of preconditions for valid application of differential equations to a phenomenon; they show that none of them are fulfilled in the case of the papers in question (e.g., no argument for the specific form of the differential equations presented, no data presented to fit the parameters). In a long response, Brown disassociates herself from the positivity ratio and from the mathematical modeling (which she attributes to Losada, who "chose not to respond"); but she adheres to her underlying psychological "broaden-and-build theory of positive emotions," for which she claims growing empirical evidence. Meanwhile, 8 years have passed since publication of the Fredrickson and Losada and paper, and 14 since the first questionable Losada paper. I would like to think that there are some psychologists versed in enough mathematics, or enough mathematicians interested enough in psychology, that such pseudoapplications could not endure unquestioned for so long.

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Epstein, Jerome, The Calculus Concept Inventory—Measurement of the effect of teaching methodology in mathematics, *Notices of the American Mathematical Society* 60 (8) (September 2013) 1018–1026, http://www.ams.org/journals/notices/201308/rnoti-p1018.pdf, DOI: http://dx.doi.org/10.1090/noti1033.

"The Calculus Concept Inventory (CCI) is a test of conceptual understanding (and only that—there is essentially no computation) of the most basic principles of differential calculus." The idea for it was based on similar concept tests in physics; more than 20 years ago, those test showed that students finishing basic college physics had little conceptual understanding of mechanics or force. Author Epstein reports similar results for the CCI, whose calculus questions are at the level of "point to your foot." Giving the CCI in other countries produced similar results except in Shanghai, where results were two standard deviations above U.S. results. The main point of the article is that while a semester of "traditional" calculus instruction has no effect on scores(!), a semester with a teaching methodology of "interactive engagement (IE)" makes a large positive difference. Other factors (class size, instructor experience, textbook, hours in class, student math background) make no difference. Instead, the key ingredient of IE appears to be *immediate* feedback from peers or instructor. Well, that makes sense: The student actually has to do something—in the moment and on the spot—on which the feedback is to be based. Beware: The concept tests in physics are now used as measures not only of learning but also of teaching effectiveness. (The CCI, as well as a Basic Skills Inventory test, are available, subject to a nondisclosure agreement, from Prof. Epstein by request to jerepst@att.net or jepstein@poly.edu.)

Cucker, Felipe, *Manifold Mirrors: The Crossing Paths of the Arts and Mathematics*, Cambridge University Press, 2013; x + 415 pp, $90, $29.99 (P). ISBN 978-0-521-42963-4, 978-0-521-72876-8.

This book begins with "appetizer" anecdotes from the lives of artists of varying eras and traditions. The anecdotes suggest a common drive toward seeking a "sense of order," of "following a pattern, an underlying law." The author follows with chapters on geometry, plane isometries, tessellations, homotheties and similarities, symmetries in music, perspective drawing, projections, anamorphic artworks, and isometries and tessellations in non-euclidean geometries. There are theorems, proofs, and mathematical symbolism. What makes the book particularly notable—and irresistible to dip into—are its numerous full-color reproductions of paintings, photos, and carpets to illustrate the mathematical concepts discussed. Author Cucker finds an approach–avoidance interplay in the use of symmetry in art, leading at times to deliberately broken symmetry, as artists and art-appreciators seek a sense for order that is "without the boredom of undue repetition." Symmetry is a "raw material," providing a major source for that order; but its rules can be bent as desired.

Suri, Manil, How to fall in love with math, *New York Times* (16 September 2013) A23, http://www.nytimes.com/2013/09/16/opinion/how-to-fall-in-love-with-math.html.

"[H]uman beings are wired for mathematics. . . . At some level, perhaps we all crave it." In this opinion piece, Suri plumps for "math appreciation"; just as one can appreciate art or music without being able to paint or play an instrument, one can appreciate "many profound mathematical ideas" without advanced math "skills." He cites as examples regular polygons homing in on a circle, "the eye candy of fractal images," and the "chain reaction" / "Big Bang" of discovering that consecutive integers march to infinity.

Knight, David, Holy logic: Computer scientists "prove" that God exists, *Spiegel Online International* (22 October 2013), http://www.spiegel.de/international/germany/scientists-use-computer-to-mathematically-prove-goedel-god-theorem-a-928668.html.

Formalization in modal logic shows, with the aid of a computer, that an argument of Kurt Gödel's for the existence of God is mathematically sound; however, it shows merely that the conclusion follows from the axioms that he set out. There is great irony here: that the man who demonstrated the limits of proof about mathematics (not all true statements can be proven in first-order logic) could nevertheless reach so grand a conclusion. Continuing that irony, I am tempted to wish: May God save us from such "proofs"!

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Here is a cool identity to celebrate the new year.

$$\left(\sqrt[3]{\sqrt{2014}+1}-\sqrt[3]{\sqrt{2014}-1}+\sqrt[6]{2013}\right)^3 + \left(\sqrt[3]{\sqrt{2014}+1}-\sqrt[3]{\sqrt{2014}-1}-\sqrt[6]{2013}\right)^3 = 4$$

Contributed by Dongvu Tonien, Australian National University

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TITLES

PROBLEMS

The letters P, Q, and S refer to Proposals, Quickies, and Solutions, respectively; page numbers appear in parentheses. For example, P1911 (63) refers to Proposal 1911, which appears on page 63.

Solution to puzzle on page 371:

P	U	L	P	S	■	■	■	W	T	S	■	M	A	A
E	P	E	E	S	■	C	R	O	A	T	■	I	N	S
G	R	A	P	H	T	H	E	O	R	Y	■	N	A	P
L	I	P	S	■	H	A	N	D	E	L	■	O	L	I
E	S	T	■	M	A	G	■	S	S	E	■	R	Y	R
G	E	O	M	E	T	R	Y	■	■	■	S	A	S	E
■	■	■	E	L	S	I	E	■	C	A	P	R	I	S
■	C	O	M	B	I	N	A	T	O	R	I	C	S	■
C	A	V	E	A	T	■	S	H	O	R	E	■	■	■
A	L	E	S	■	■	■	T	O	P	O	L	O	G	Y
V	C	R	■	A	M	S	■	R	E	W	■	P	R	E
E	U	R	■	L	E	T	T	E	R	■	G	T	O	S
A	L	A	■	P	R	O	B	A	B	I	L	I	T	Y
R	U	T	■	H	I	N	D	U	■	D	E	N	T	E
T	S	E	■	A	T	E	■	■	■	L	E	G	O	S

NEWS AND LETTERS

Letter to the Editor

To the Editor,

The main result in Victor Oxman's note, "Two Cevians Intersecting on an Angle Bisector," in the June, 2012 issue of this MAGAZINE, had been already obtained, in a stronger form, by V. Nicula and C. Pohoaţă in 2009 and appeared in [**1**]. The results were later strengthened further by S. Abu-Saymeh and M. Hajja in 2010, and appeared in [**2**].

MOWAFFAQ HAJJA
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Victor Oxman writes: "I really did not know about them before writing my article, otherwise I would have mentioned them, of course." Alas, neither did we. Oxman's proof is a gem, as is the result itself. We are glad to recognize the priority of Nicula, Pohoaţă, Abu-Saymeh, and Hajja.

EDITOR

REFERENCES

1. V. Nicula and C. Pohoaţă, A stronger form of the Steiner-Lehmus theorem, *J. Geom. Graphics* **13** (2009) 25–27.
2. S. Abu-Saymeh and M. Hajja, More on the Steiner-Lehmus theorem, *J. Geom. Graphics* **14** (2010) 127–133.
3. V. Oxman, Two cevians intersecting on an angle bisector, *Mathematics Magazine* **85** (2012) 213–215.

Math. Mag. **86** (2013) 398. doi:10.4169/math.mag.86.5.398.

New Editor of *Mathematics Magazine*

We are pleased to announce that Michael A. Jones has been elected by the Governors to be Editor of this MAGAZINE for the term 2015–2019. As Editor-elect, he will be responsible for reviewing new submissions during the calendar year 2014.

Mike is an Associate Editor at the American Mathematical Society's *Mathematical Reviews* in Ann Arbor, MI. He earned his Ph.D. in game theory under Donald Saari at Northwestern University in 1994. After positions at the U.S. Military Academy at West Point and Loyola University in Chicago, he spent 10 years at Montclair State University. He served on the editorial boards of the *College Mathematics Journal* from 2005 to 2013 and of the MAA's Spectrum Book Series from 2003 to 2010. His research interests are loosely described as the application of mathematics to the social sciences, but he wanders into other areas of pure and applied mathematics if given the chance. He also writes expository articles and articles to highlight the application of mathematics to the social sciences for high school and undergraduate classrooms. His article on Chutes and Ladders, co-authored with undergraduate students Leslie Cheteyan and Stewart Hengevelde, was awarded the 2012 Pólya Award. He has been a member of the MAA since 1987, when he received a student membership from Santa Clara University. He looks forward to following in the footsteps of Gerald L. Alexanderson (his calculus professor and undergraduate advisor) and Frank A. Farris (his advanced calculus professor), as both are former editors of the MAGAZINE.

Submissions to the Magazine

Beginning January 1, 2014, items proposed for inclusion in the MAGAZINE should be submitted using the MAGAZINE'S Editorial Manager System. Prepare your article in the form of a pdf, dvi, or doc file, and then follow the instructions at the MAGAZINE'S website. Ordinarily manuscripts should not include the authors' names or affiliations. Questions concerning new submissions may be addressed to the Editor-elect at mathmag-editor-elect@maa.org. Other questions and correspondence can be sent to the Editor at mathmag@maa.org.

Contributions to the Problems Section should be sent to Bernardo M. Ábrego, Problems Editor, Department of Mathematics, California State University, Northridge, 18111 Nordhoff St, Northridge, CA 91330-8313 or by e-mail to mathmagproblems@csun.edu.

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